Efficient and Incentive Compatible Mediation: An Ordinal Mechanism Design Approach*

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Abstract

Mediation is an alternative dispute resolution method, which has gained increasing popularity over the last few decades and become a multi-billion-dollar industry. When two or more parties are in a disagreement, they can take the case to a court and let the judge make a binding final decision. Alternatively, the disputing parties can get assistance from an experienced, neutral third party, i.e., a mediator, who facilitates the negotiation and help them voluntarily reach an agreement short of litigation. The emphasis in mediation is not upon who is right or wrong, but rather on exploring mutually satisfactory solutions. Employment disputes, patent/copyright violations, construction disputes, and family disputes are some of the most common mediated disputes. The rising popularity of mediation is often attributed to the increasing workload of courts, its cost effectiveness and speed relative to litigation, and disputants' desire to have control over the final decision. Many traditional "cardinal" settings of bargaining and mechanism design, starting with the seminal work of Myerson and Satterhwaite (1983), have shown the incompatibility between efficiency and incentives, even in Bayesian sense. This paper uses an "ordinal" market/mechanism design approach, where the mediator seeks a resolution over (at least) two issues in which negotiators have diametrically opposed rankings over the alternatives. Each negotiator has private information about her own ranking of the outside option, e.g., the point beyond which the negotiator would rather take the case to the court. We construct a simple theoretical framework that is rich and practical enough allowing for optimal mechanisms that the mediators can use for efficient resolution of disputes. We propose and characterize the class of strategy-proof, efficient, and individually rational mediation mechanisms. A central member of this class, the compromise rule stands out as the unique strategy-proof, efficient, and individually rational mechanism that minimizes rank variance. We also provide analogous mechanisms when the issues consist of a continuum of alternatives.

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"*Mediation has rapidly become, with precious little fanfare, the ocean we swim in and the air we breathe. It would now be hard to imagine a world where it wasn't.*" Jim Melamed[1]

1. Introduction

The best-seller book, "Getting to Yes," by Roger Fisher and William Ury is arguably one of, if not the most famous, works on the topic of negotiation. They identify conflict as a growth industry, and the last few decades proved them right. Courts in all US states offer some form of ADR (alternative dispute resolution) for the cases filed in state courts. 17 states require mandatory mediation: 11% of civil cases in Northern California courts in 2011, 35.6% of civil and 21.6% of divorce cases in New York state courts in 2016 have been mediated.[2] Total value of mediated cases in UK is estimated to be £10.5bn. in 2011, excluding mega-cases, family and community disputes.[3] In addition to face-to-face mediation practices, online dispute resolution, aiming to resolve disputes that arise online, has also gained increasing popularity over the last decade. These are small disputes in size but large in number. Dispute resolution centers of E-bay, PayPal, Uber and Amazon tackles more than a billion disputes a year. Many online dispute resolution web sites use automated mechanisms to help parties resolve their disputes. Empirical studies and mediation program evaluations suggest 60-90% success rate, 90-95% satisfaction by the disputants and higher rate of compliance relative to court-imposed orders.

Unlike litigation and arbitration, mediation does not search for truth, rather searches for satisfaction. In mediation a neutral third party facilitates communication and negotiation, promotes exploration of mutually acceptable alternatives. Namely, the emphasis is not on who is right or wrong, but rather upon establishing a workable solution that meets the participants' needs. Disputants prefer mediation over it's alternatives because it is cost effective. According to Hadfield (2000) it costs a minimum of $100,000 to litigate a straightforward business claim in the US, whereas a mediation session varies from few hours to a day and even the most reputable mediators charge around $10,000 - $15,000 for a day. In addition, disputants do not have to pay any fees for expert, witness, document preparation, investigation or paralegal, which would easily pile up the costs. Airline companies and hospitals, for example, prefer mediation because mediation sessions are private and confidential. It is impossible to discuss a legally "irrelevant" issue in litigation/arbitration and some disputes are not just about money or being right: A plaintiff,

[1]Founder and CEO of Mediate.com, recipient of American Bar Association Institutional Problem Solver Award.

[2]Sources: dispute resolution centers of State of New York and California.

[3]The Seventh Mediation Audit, Centre for effective dispute resolution.

for example, would be suing her employer for sexual harassment or discrimination, and reinstatement or a good reference letter would be more important for her than compensation. On the other hand, in mediation parties can discuss and negotiate issues that are not directly linked to the case. This infinite flexibility of bringing any issue on the table can be used to transform a competitive, "zero-sum" negotiation problem into a "multi-issue" negotiation problem that enlarges the set of acceptable outcomes.

Market design has been fruitful in many applications, most notably in auctions and matching theory. The goal of this paper is to offer a first market design setting to analyze dispute resolution via mediation, which is simple enough to be practically relevant while maintaining the informational richness and complexities faced in practical disputes. To this end, our modeling significantly departs from the traditional mechanism design approach to bargaining that builds on the seminal work of Myerson and Satterthwaite (1983) in the context of bilateral trade over a good for which traders have private valuations each drawn from pre-specified distributions and commonly known utility functions. This type of "cardinal approach" has however been the subject of the famous Wilson critique for it lacks "detail-freeness" and does not provide robust incentives to participants. In a similar vein, Ausubel, Crampton, and Denecker voice a similar concern:

> "*... Despite these virtues, mechanism design has two weaknesses. First, the mechanisms depend in complex ways on the traders' beliefs and utility functions, which are assumed to be common knowledge. Second, it allows too much commitment. In practice, bargainers use simple trading rules—such as a sequence of offers and counteroffers—that do not depend on beliefs or utility functions.*" Handbook of Game Theory

The ordinal approach, whereby the designer elicits only ordinal preference information, has already lead to quite notable success in applications of matching and assignment such as medical residency, school choice, kidney exchange, and course assignment, where a plethora of strategy-proof and efficient mechanisms have been obtained, extensively studied, and even adopted in practice.

Our model assumes that two negotiators are in a dispute and aim to reach a resolution through a mediator. There is a main issue, issue X, consisting of a finite number of alternatives, which is relevant for both parties' welfare.[4] The negotiators have diametrically opposed preferences over alternatives in the sense that if one negotiator prefers one alternative over another, then the other negotiator has exactly opposite ranking of the two alternatives. However, not all alternatives are acceptable for any given negotiator. When offered one such alternative for her, a negotiator rejects the mediator's proposal and

[4]We later relax the finiteness and discreteness assumptions on X.

pursues alternative ways of resolution, e.g., litigation. We capture such circumstances by assuming an outside option whose ranking is each negotiator's private information. The mediator's objective is to truthfully elicit negotiators' private information about the position of their outside options and propose an efficient and mutually acceptable, i.e., individually rational, outcome.

We first show that if there is a single issue, i.e., no other issues than issue X, then there is no strategy-proof, efficient, and strategy-proof mechanism. Furthermore, we show that this impossibility extends to multiple issues if each issue has an outside option similarly defined, i.e., in each issue each negotiator has an outside option whose ranking is her private information. This motivates the need for a setting that asymmetrically treats different issues: Consider a second issue, issue Y, where the outside option is the least preferred alternative for both negotiators. This asymmetric treatment of the outside options can be motivated by various employment, family, construction or patent/copyright infringement disputes. Litigation is naturally the default option when the issue is compensation or division of property. If parties expect litigation to be a very long and costly process, then any division of surplus would be efficient. In that regard, compensation would be considered as issue Y. Although money is an important issue in disputes, it is rarely the only issue (Malhotra and Bazerman, 2008). Disagreements over change orders, extra work or the scope of work in construction disputes, and child custody or visitation in family disputes would be the additional issues, i.,e, issue X, where the parties are uncertain about their opponents' acceptable alternatives. Alternatively, in a bilateral negotiation between a worker and an employer, issue Y would represent wage, where both parties' outside options are inferior to any division of the surplus that they are negotiating, and the location of the office would represent issue X.

In the two-issue mediation problem, the mediator recommends a bundle (x, y) of outcomes from $X \times Y$. A mediation rule/mechanism is a systematic way of choosing an outcome for any given preferences of the negotiators. Since the mediator asks negotiators to report their outside options over alternatives in issue X (recall that there is no uncertainty regarding negotiators' preferences over alternatives in issue Y), one needs to invoke extension mappings to obtain the possible set of negotiators' underlying preferences over bundles. Alternatively, it is conceivable that the mediator elicits preferences over bundles of alternatives. This approach, which we do not pursue, however, has two drawbacks: First, the number of bundles to rank increases quadratically with the number of alternatives in each issue, which in turn makes asking for full-fledged rankings over bundles highly impractical. Second, a similar impossibility to the single-issue mediation would arise in this case.

In this paper we ask if there is an impartial and dominant strategy incentive compatible, i.e., strategy-proof, way of soliciting true preferences so that mediation outcomes are efficient and individually rational. A sufficient and almost necessary condition for obtaining a positive answer to this question is the so-called "logrolling (quid pro qu)" condition on negotiators' preferences. This assumption imposes a form of substitutability between issues X and Y. More specifically, logrolling requires preferences to be rich enough such that a negotiator is able to compromise issue X for a more preferred alternative in issue Y, e.g., for a given (x, y) bundle, there exists a (weakly) more preferred bundle which involves getting a worse alternative in X combined with a better alternative in Y. In the continuous version of our model, we show that many commonly used utility functions satisfy this assumption.

Our main result is a complete characterization of the class of strategy-proof, efficient, and individually rational mediation rules. These rules operate through an exogenously specified precedence order over a set of special bundles, which we call as the logrolling bundles and always make selections among these bundles. The logrolling bundles form a simple lattice structure with respect to the negotiators' preferences: given any set of mutually acceptable alternatives, for each negotiator there is always an optimal-logrolling bundle that she prefers over all other acceptable bundles; this bundle is the pessimal-logrolling bundle for the opposite negotiator. The characterized class of rules nest interesting extremal members. When the precedence order coincides with the preference ranking of a given negotiator over the logrolling bundles, we obtain the corresponding negotiator-optimal rule.

In keeping with our main objective of finding impartial mediation rules, we search for members of this class of rules that satisfy sensible fairness criteria. To this end, we define the "rank variance" of an outcome as the sum of the square of each negotiator's ranking of each alternative in each issue. It turns out there is a unique member of the family of strategy-proof, efficient, and individually rational mediation rules that minimizes rank variance. This is the so-called "Compromise" rule, which recommends the median logrolling bundle when it is mutually acceptable, or the closest mutually acceptable logrolling bundle to it when it is not mutually acceptable. This rule is intuitive and simple enough to be used as a standardized protocol for finding the middle ground between disputing parties in practice.

Related Literature

Our paper and modeling approach connects and spans four different types of literature:

1) Bargaining and Mechanism Design: Mediation is a part of the bargaining literature, which is primarily based on the cardinal approach discussed above. The more broadly-

defined mechanism design approach to bargaining in the presence of private outside options, started with the classic paper by Myerson and Satterthwaite (1983) [MS henceforth], has generally emphasized the difficulty/impossibility of reaching efficient outcomes even in Bayesian settings let alone dominant strategies. Specifically, for the mediation context, there are very few papers: Bester and Warneryd (2006) show that the news are even worse than MS in this setting. The MS result depends on there being a positive probability of trade being inefficient ex post. Bester and Warneryd (2006), in a model featuring continuum of types, show that asymmetric information about relative strengths as an outside option in a conflict may cause agreement to be impossible even if the agreement is always efficient. In their model, conflict shrinks the pie and thus agreement on a peaceful settlement is always ex post efficient. Following Bester and Warneryd (2006), Horner et al (2015) compare the optimal mechanisms, with two types of negotiators, under arbitration, mediation and unmediated communication. For both models, there is no ex post efficient and Bayesian incentive compatible mechanism. Namely, optimal mechanism is necessarily inefficient.

In our model, we adopt an ordinal mechanism design approach in the sense that negotiators rank finitely many available options in opposite ways, which is common knowledge, but the outside option of each negotiator is her private information as in the Compte and Jehiel (2007) model, which adopts a cardinal utility approach much like the rest of this literature. In our benchmark model of single-issue mediation, a conflict situation which is defined as the mediators recommending negotiators to exercise their outside options, is also ex post inefficient so long as the mediators have a mutually acceptable outcome (clearly, when there is no mutually acceptable outcome, mediation is hopeless). Also, for the second issue Y, the conflict situation (outside option) is always inefficient in our model.

By contrast to this literature, our ordinal approach together with our modeling specifications enables us to obtain positive results: Indeed, we are able to achieve ex post efficiency in dominant strategies and argue that the proposed rules can potentially be convenient and simple enough to use in practice.

2) Political Economy: Our benchmark model (but not the main, two-issue model where preferences over bundles are not necessarily single-peaked) resembles a voting model with single-peaked preferences where a number of voters have single-peaked preferences over the single-dimensional political spectrum and a voting rule aggregates individual preferences (Black 1948, Moulin 1980, Barbera, Gul and Stachetti 1991, and Ching 1997). In this type of models, the famous median voter theorem states that majority-rule voting system will select the outcome most preferred by the median voter.

In our model, preferences can also be thought to be single-peaked with each negotiator preferring the opposite extremes of the spectrum. There are a number of differences in our model from a voting model. In a voting model, there are several voters whose bliss point (peak value) is their private information. In our model, peaks are publicly known. What is private information is the two negotiators' outside options, which do not have any analogues in a voting model. Because of this difference, the voting model admits a class of strategy-proof rules for which efficiency and individual rationality vacuously hold. In our setup, however, such rules are either inefficient or violate individual rationality. In our setup, even dictatorship rules, despite being efficient, violate individual rationality.

In our benchmark model with single issue, there is no strategy-proof, efficient and individually rational rule. This in turn motivates to consider multi-issue models where single-peakedness does not necessarily hold and there is asymmetry in terms of outside options are treated for each issue. In fairness, there are also multi-dimensional voting models where people vote on multiple issues but this literature also concludes that strategy-proofness effectively requires each dimension to be treated independently for other. In our setup, by contrast we exploit a kind of exchangeability between the two issues, together with an asymmetric treatment of outside options, to arrive at strategy-proof rules

A logically independent but similar result that we find to the median voter theorem is that although the family of strategy-proof, efficient and individually rational rules we characterize are much different than those strategy-proof rules characterized generalized Condorcet rules, our family also nests a central rule dubbed the compromise rule that chooses the "median bundle" when preferences of the negotiators are symmetric and tries to choose outcomes as close to the median bundle as possible. In our setup, however, the class of rules need not even include the impartial median-type rule. Indeed, we also identify rules that may also be partial toward either negotiator.

3) Matching/Assignment: Matching models and applications have championed the ordinal mechanism design approach (see, for example, Gale and Shapley 1962, Shapley and Shubik 1971, Crés and Moulin 2001, and especially recent applications of ordinal assignment mechanisms Balinski and Sönmez 1998, Abdulkadiroglu and Sönmez 2003, Roth, Sönmez and Ünver 2005.) Ordinal rankings over objects together with an outside option is a common feature of matching/assignment models. Given that both negotiators end up consuming the same bundle, our model with ordinal preferences can be though to be a "public good" assignment version of a matching problem. This connection to matching is important in two regards: 1. Ordinal mechanisms may be more practical and convenient than cardinal ones as supported by experimental work. In this regard ordinal mechanisms coupled with strategy-proofness can help avoid the Wilson's critique often imposed on the "cardinal/Bayesian" mechanism design approach.

A second connection that surfaces to matching type models as a result of our analysis is that we find that the class of strategy-proof, efficient and individually rational rules also contain the negotiator-optimal rules, much in the same spirit as the proposing-side optimal deferred acceptance mechanisms or the buyer/seller optimal core assignments in the Shapley-Shubik assignment game.

4) Non-dictatorial strategy-proof mechanisms escaping the Arrow - Gibbard - Satterthwaite impossibilities: With the hope of arriving at possibility results, there is a tradition of identifying strategy-proof rules in restricted economic environments: see, for example, Vickrey (1961), Groves (1973), Clarke (1971) [VCG] for public goods and private assignment with transfers, uniform rule (Benassy 1982, Sprumont 1991) for division of divisible private good under single-peaked preferences, generalized median-voters (Moulin 1980), proportional-budget exchange rules (Barbera and Jackson 1995) that allow for trading from a finite number of pre-specified proportions (budget sets), deferred acceptance (Gale and Shapley, 1962) and top trading cycles (David Gale, 1974 and Abdulkadiroglu and Sönmez, 2003); hierarchical exchange and brokerage (Papai 2001 and Pycia and Ünver 2015). We also add to this literature in the sense that one may draw a conceptual parallel with the VCG mechanisms, though our rules look nothing like the above rules. In the VCG model, preferences over objects are private info and the preferences over money is common knowledge. This is much like negotiator's preferences over issue X versus issue Y. This connection is only superficial since VCG mechanisms are cardinal, and assignments and transfers depend on reported utilities.

2. The Environment

We begin to describe the environment with a simple example and a short discussion about why the assumption of diametrically opposed preferences is without loss of generality.

A simple example: single-issue mediation

Negotiators 1 and 2 are in dispute over a single issue that is important for both. Let x_1 and x_2 denote the available alternatives (solutions) for the dispute. The negotiators are also entitled to the outside option, o, in case one or both of them reject to accept one of the alternatives. Therefore, the set $X = \{x_1, x_2, o\}$ denotes the set of all possible outcomes of the dispute.

It is common knowledge that negotiator 1 (strictly) prefers alternative x_1 to x_2 and negotiator 2 prefers x_2 to x_1. That is, the negotiators have diametrically opposed preferences over the alternatives x_1 and x_2. The ranking of the outside option, however, is the negotiators' private information. Therefore, each negotiator has two types[5]:

[5]We suppose, without loss of generality, that there is at least one acceptable alternative for each negotiator.

$\theta_1^{x_1}$	$\theta_1^{x_2}$
x_1	x_1
o	x_2
x_2	o

$\theta_2^{x_2}$	$\theta_2^{x_1}$
x_2	x_2
o	x_1
x_1	o

Consider the mediation process, denoted by f, as a mechanism with veto rights that maps the negotiators' private information to an outcome in X. Then, it would be represented by the following matrix

	$\theta_2^{x_1}$	$\theta_2^{x_2}$
$\theta_1^{x_1}$	$f_{1,1}$	$f_{1,2}$
$\theta_1^{x_2}$	$f_{2,1}$	$f_{2,2}$

where $f_{\ell,j} \in X$ for all $\ell, j \in \{1, 2\}$.

We can assign $f_{1,2} = o$, without loss of generality, because there is no mutually acceptable alternative when the negotiators' types are $\theta_1^{x_1}$ and $\theta_2^{x_2}$, and thus, the outside option o is effectively the only result in all voluntary mediation processes. If the outcomes of the mediation process are (Pareto) efficient, then $f_{1,1}$ should be x_1 or x_2. Moreover, if the process produces individually rational outcomes, then we must have $f_{1,1} = x_1$. Likewise, an efficient and individually rational mediation process suggests $f_{2,2} = x_2$ and $f_{2,1} \in \{x_1, x_2\}$.

Therefore, we can construct several efficient and individually rational mechanisms for this simple example. However, none of these processes are immune to strategic manipulation (strategy-proofness). To prove this point, suppose that $f_{2,1} = x_1$. In this case, type $\theta_2^{x_1}$ of negotiator 2 would deviate and declare his type as $\theta_2^{x_2}$ to obtain x_2, contradicting with strategy-proofness. Alternatively, if $f_{2,1} \neq x_1$, then type $\theta_1^{x_2}$ of negotiator 1 would deviate and declare his type as $\theta_1^{x_1}$ to obtain x_1, again contradicting with strategy-proofness.

It is easy to extend this example to the case with more than two alternatives, and so extrapolate that there exists no efficient, individually rational and strategy-proof single-issue mediation process.[6]

Modeling conflicting preferences

Using diametrically opposed preferences over alternatives, when describing a dispute, is intuitive because it resembles the standard bargaining problem, which is modeled as a zero sum game, and unavoidable when the number of available alternatives is just two.

[6] However, there are efficient, individually rational and Bayesian incentive compatible mediation rules when negotiators are sufficiently risk averse (Kesten and Ozyurt, 2018).

However, intuition suggests that many other preference profiles, which are not diametrically opposed, would also depict a dispute when there are more than two alternatives. Consider, for example, the case where the set of available alternatives (other than the outside option) is $A = \{x_1, x_2, x_3, x_4, x_5\}$ and the negotiators' preferences are

θ_1	θ_2
x_1	x_3
x_2	x_5
x_3	x_4
x_4	x_2
x_5	x_1

These preferences are not diametrically opposed but they are certainly conflicting—to some degree—as the agents cannot agree on their best alternative. Notice, however, that alternatives x_4 and x_5 are (Pareto) dominated by x_3, and so, if selecting an efficient outcome by the mediation protocol is desired, then the presence of these two alternatives is irrelevant for the problem. Knowing whether or not these two alternatives are acceptable, i.e., better than the outside option, is also an "irrelevant" piece of information because these alternatives are acceptable by a negotiator whenever x_3 is acceptable. Thus, this particular dispute problem can be transformed into a simplified and "outcome equivalent" version where the only available alternatives are x_1, x_2 and x_3 and the negotiators' preferences over these three are diametrically opposed. We can generalize this observation for any (discrete) set of alternatives and for any preference profiles, where negotiators cannot agree upon their first best.

Let A be non-empty set of available alternatives and Θ be the set of all complete, transitive and antisymmetric preference relations on A. Define $max(\theta)$ to be the maximal element of the preference ordering $\theta \in \Theta$, namely if $x^\star = max(\theta)$, then $x^\star \ \theta \ x$ for all $x \in A \setminus \{x^\star\}$. Therefore, a ***two-person, single-issue dispute*** (dispute in short) problem is a list $D = (\theta_1, \theta_2, A)$ where $\theta_i \in \Theta$ for $i = 1, 2$ and $max(\theta_1) \neq max(\theta_2)$.

For any non-empty subset $\widetilde{A}$ of A, let $\theta|_{\widetilde{A}}$ denote the restriction of the preference ordering $\theta \in \Theta$ on $\widetilde{A}$. Therefore, define $\widetilde{D} = (\widetilde{\theta}_1, \widetilde{\theta_2}, \widetilde{A})$ to be a dispute reduced from $D = (\theta_1, \theta_2, A)$ whenever $\widetilde{A} \subseteq A$ and $\widetilde{\theta}_i = \theta_i|_{\widetilde{A}}$ for $i = 1, 2$.

Proposition 1. *By deleting all the Pareto inefficient alternatives, any two-person, single-issue dispute problem D can be reduced into a two-person, single-issue dispute problem $\widetilde{D}$ where the negotiators preferences are diametrically opposed.*

All proofs are deferred to Appendix. A similar result, which we omit to avoid repetition, holds for two-person, multi-issue mediation problems whenever preferences over bundles satisfy monotonicity.[7]

[7]See next section for the formal definition of monotonicity.

3. The Main Model: Multi-Issue Mediation

This section proves that we may escape from the impossibility result akin to Myerson and Satterthwaite (1983) if the negotiators are in dispute over multiple issues and two issues are enough to make our point. There are two agents, $\mathrm{I} = \{1, 2\}$, in a dispute who aim to reach a resolution through mediation. Without loss of generality, there are two **issues** that are important for the negotiators.[8] Let the sets $X = \{x_1, ..., x_m, o_X\}$ and $Y = \{y_1, ..., y_m, o_Y\}$ denote the finite sets of potential **outcomes** for each issue. The sets $X \setminus \{o_X\}$ and $Y \setminus \{o_Y\}$ are the available **alternatives**. The negotiators are entitled to an **outside option** (disagreement point) for each issue, o_X and o_Y, in case one or both of them reject to accept an alternative that is available for that issue. Negotiators have at least two available alternatives for each issue, and so $m \geq 2$.[9]

Preferences over Outcomes: The negotiators' preferences over outcomes for each individual issue satisfy the following three condition:

1. The negotiators' preferences over alternatives (not including the outside option) for each individual issue are diametrically opposed and public information.

2. Both negotiators' rankings of the outside option (relative to other alternatives) are private information in one of the issues.

3. It is public information that both negotiators rank the outside option as their worst outcome in one of the issues.

More formally, for any issue $Z \in \{X, Y\}$, where $Z = \{z_1, ..., z_m, o_Z\}$, let Θ_i^Z denote the set of all complete, transitive and antisymmetric preference relations of negotiator $i \in \mathrm{I}$ over issue Z and θ_i^Z denote an ordinary element of the set Θ_i^Z. It is public information that $z_k \; \theta_1^Z \; z_{k+1}$ and $z_{k+1} \; \theta_2^Z \; z_k$ for all $k = 1, ..., m-1$. Namely, the negotiators' preferences over the alternatives for each issue are diametrically opposed (the first condition). The ranking of the outside option in issue X, o_X, is the negotiators' private information (the second condition). Finally, it is common knowledge that $y \; \theta_i^Y \; o_Y$ for all i and $y \in Y \setminus \{o_Y\}$ (the third condition). Therefore, the set of acceptable alternatives for issue X is privately known by the negotiators, and it is unknown to them whether there is a mutually acceptable alternative for that issue. However, all alternatives in issue Y are acceptable by both negotiators and efficient. Note that there is a unique preference ordering in Θ_i^Y

[8]The case with more than two issues is discussed in Section 6.

[9]The number of alternatives in issue X, $\#X$, must be greater than or equal to the number of alternatives in issue Y, $\#Y$, for a possibility result. However, the assumption that each issue has the same number of alternatives is not essential for the possibility. A more detailed discussion for $\#Y > \#X$ is deferred to Section 6.

and $m+1$ orderings in Θ_i^X. Therefore, let $\Theta_i = \Theta_i^X$ denote the set of all **types** for negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ be the set of all type profiles.

This asymmetric treatment of the outside options can be motivated by various employment, family, construction or patent/copyright infringement disputes. Litigation would naturally be the default option if the issue is compensation or division of property. It usually is the case in such disputes that litigation is a very long and costly process, and so, inefficient relative to other potential divisions (alternatives). Such issues would be mapped into the issue Y in our framework. Although money is an important component in disputes, it is not the only issue: In employment disputes, for example, the quality of the reference letter that the former employer would be willing to write could be another issue for the disputants, or child custody or visitation would be the alternative issues in family disputes. Such issues, where the disputants' ranking of the outside option is not clear to all the parties, would be represented by the issue X in our setup. Nonetheless, it is natural to find examples, where the ranking of the outside option in all issues are the disputants' private information. For that reason, the symmetric treatment of the outside option is formally investigated in Section 6, where we prove that there exists no efficient, individually rational and strategy-proof mediation rules in this case.

Preferences over Bundles: A **bundle** (x, y) is a vector of outcomes, one for each issue, and the set $X \times Y$ denotes the set of all bundles. Let $\Re$ denote the set of all complete and transitive binary relations over the bundles. R is a standard element of the set $\Re$ and for any two bundles $b, b' \in X \times Y$, $b \; R \; b'$ means "b is at least as good as b'." We denote P for the strict counterpart of R.[10] We consider a restricted domain of preferences over bundles, defined through a particular extension map that satisfies three consistency axioms. An extension map is a rule Λ that assigns to every negotiator i and type $\theta_i \in \Theta_i$ a non-empty set $\Lambda(\theta_i) \subseteq \Re$ of admissible orderings over bundles.

For any negotiator i and type $\theta_i \in \Theta_i$, let $A(\theta_i) = \{x \in X | \; x \; \theta_i^X \; o_X\}$ denote the set of **acceptable** alternatives in issue X. For any type profile $(\theta_1, \theta_2) \in \Theta$, the set $A(\theta_1, \theta_2) = \{x \in X | \; x \; \theta_i^X \; o_X \text{ for all } i \in N\}$ denote the set of **mutually acceptable** alternatives in issue X. In case we need to specify a type's acceptable alternatives, we use $\theta_i^x \in \Theta_i^X$: It denotes the preference relation (type) of negotiator i in which alternative $x \in X$ is the worst acceptable alternative. Namely, for any $x' \in X \setminus \{o_X\}$, $x \; \theta_i^x \; x' \implies o_X \; \theta_i^x \; x'$.

Definition 1. *The **extension map** Λ is **regular** if the followings hold for all i, $\theta_i \in \Theta_i$ and all $R_i \in \Lambda(\theta_i)$:*

[10]That is, $b \; P \; b'$ if and only if $b \; R \; b'$ holds but $b' \; R \; b$ does not.

i. [Monotonicity] For any $x, x' \in X$ *and* $y, y' \in Y$ *with* $(x, y) \neq (x', y')$*,*

$$(x, y)\ P_i\ (x', y') \text{ whenever } \left[x\ \theta_i^X\ x' \text{ or } x = x'\right] \text{ and } \left[y\ \theta_i^Y\ y' \text{ or } y = y'\right].$$

ii. [Deal Breakers] For any $y, y' \in Y \setminus \{o_Y\}$*,*

$$(x, y)\ R_i\ (x', y') \text{ whenever } x \in A(\theta_i) \cup \{o_X\},\ x' \notin A(\theta_i) \text{ and } x \neq x'.$$

Monotonicity is a standard assumption. The second condition suggests that unacceptable alternatives in issue X are "deal-breakers" for the negotiators: regardless of the alternative in the second issue, a bundle with an unacceptable alternative is never preferred to a bundle with an acceptable alternative or to a bundle with the outside option o_X. Put differently, an alternative x' in issue X is called unacceptable whenever there is no sufficiently attractive alternative y' in issue Y that makes x' "tradable" with a better alternative x. For example, in a negotiation between a candidate and an employer, which has multiple offices in different cities/countries, the candidate would have strict locational preferences that make some alternatives unacceptable regardless of the wage that is feasible by the employer. The deal-breakers assumption is not necessary for a possibility result and this issue is further discussed in Section 6, where we provide an efficient, individually rational and strategy-proof mediation rule where preferences violate deal-breakers property.

Definition 2. *The* ***extension map*** Λ *satisfies* ***logrolling*** *if there exists a function* $t : X \to Y$ *such that for all* i*,* $\theta_i \in \Theta_i$*,* $R_i \in \Lambda(\theta_i)$ *and all* $x, x' \in A(\theta_i)$ *with* $x\ \theta_i\ x'$*, we have* $\big(x', t(x')\big)\ R_i\ \big(x, t(x)\big)$*.*

Logrolling allows "trading" of alternatives. It requires two things. First, for any two acceptable alternatives x, x' in X where x is ranked above x' for type θ_i, there must exist two alternatives y, y' in Y, where y' is ranked above y, such that (x', y') is ranked at least as high as (x, y) at all admissible orderings over bundles, $R_i \in \Lambda(\theta_i)$. Second, types must be "consistent." Namely, order reversing mapping, t, is independent of types. Logrolling implies that attractiveness of the alternatives in Y are sufficiently dispersed so that negotiators are willing to trade one acceptable alternative in issue X with another acceptable alternative in X. Logrolling rules out lexicographic preferences and many standard utility functions satisfy it. We discuss this point later in detail. Note that logrolling is a well-defined concept only if the number of alternatives in issue Y is greater than (or equal to) the number of alternatives in issue X.

Example 1 (logrolling): Suppose that $X = \{x_1, x_2, x_3, o_X\}$ and $Y = \{y_1, y_2, y_3, o_Y\}$. Because the number of alternatives in issues X and Y are equal, there is a unique one-to-one function t, where $t(x_k) = y_{4-k}$ for $k = 1, 2, 3$, which satisfies the requirements of Definition 2. Logrolling implies that the type $\theta_1^{x_3}$ of negotiator 1 who deems all three alternatives in issue X acceptable, i.e., $x_1\ \theta_1^{x_3}\ x_2\ \theta_1^{x_3}\ x_3\ \theta_1^{x_3}\ o_X$, will rank (x_3, y_1) at least as high as the bundle (x_2, y_2) and rank (x_2, y_2) at least as high as the bundle (x_1, y_3) for all admissible orderings $R \in \Lambda(\theta_1^{x_3})$. The consistency of the mapping t over the types implies, for example, that type $\theta_1^{x_2}$ of negotiator 1 who deems only x_1 and x_2 acceptable, i.e., $x_1\ \theta_1^{x_2}\ x_2\ \theta_1^{x_2}\ o_X\ \theta_1^{x_2}\ x_3$, will rank (x_2, y_2) at least as high as the bundle (x_1, y_3). Logrolling imposes no restriction on admissible orderings $R \in \Lambda(\theta_1^{x_2})$ regarding how they rank the bundle (x_3, y_1) relative to the bundles (x_2, y_2) and (x_1, y_3).

Remark: For the rest of the paper, we let $\mathbf{B}$ denote the **set of logrolling bundles**. Namely, $\mathbf{B} = \left\{(x_k, y_{m+1-k}) \in X \times Y \middle| k = 1, ..., m\right\}$.

Direct Mechanisms with Veto Rights: Mediation would be a very complicated, multi-stage game between the negotiators and the mediator. The mediation protocol, whatever the details are, produces proposals for agreement that are always subject to unanimous approval by the negotiators. That is, before finalizing the protocol, each negotiator has the right to veto the proposal and the option to receive the outside options.

A version of the revelation principle that we prove in the appendix guarantees that we can stipulate the following direct mechanism with veto rights without loss of generality, when representing mediation. The direct mediation mechanism consists of two stages: an *announcement* stage and a *ratification* stage; and it is characterized by the mediation rule $f : \Theta \to X \times Y$. After being informed of its type, each negotiator i privately reports his type, $\widehat{\theta}_i$, to the mediator, who then proposes $f(\widehat{\theta}_1, \widehat{\theta}_2) \in X \times Y$. In the ratification stage, each party simultaneously decides whether to accept or veto the proposed bundle. In case both negotiators accept the proposed bundle, then it becomes the final outcome. In case one or both negotiators veto the proposal, each party gets the outside option for both issues, i.e., (o_X, o_Y). Such direct mechanisms will be called direct truthful mechanisms with veto rights.

Definition 3. *The mediation rule f is **strategy-proof** if for all i and all $\theta_i \in \Theta_i$, $f(\theta_i, \theta_{-i})\ R_i\ f(\theta_i', \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.*

Definition 4. *The mediation rule f is **individually rational** if for all i and all $(\theta_i, \theta_{-i}) \in \Theta$, $f(\theta_i, \theta_{-i})\ R_i\ (o_X, o_Y)$ for all $R_i \in \Lambda(\theta_i)$.*

Definition 5. *The mediation rule f is **efficient** if there exists no $(\theta_i, \theta_{-i}) \in \Theta$ and*

$(x', y') \in X \times Y$ *such that* $(x', y')\ R_i\ f(\theta_i, \theta_{-i})$ *for all* $R_i \in \Lambda(\theta_i)$ *and all* $i \in I$, *and for at least one* $i \in I$, $(x', y')\ P_i\ f(\theta_i, \theta_{-i})$ *for some* $R_i \in \Lambda(\theta_i)$.

We seek direct mechanisms with veto rights in which, it is a *dominant strategy equilibrium* to report the true private information at the announcement stage, and in which, in equilibrium, proposals are not vetoed. It immediately follows from the definitions that such an equilibrium exists if and only if the mediation rule f is strategy-proof and (ex-post) individually rational.[11]

4. Main Results: Strategy-Proof Mediation

For convenience, we present a mediation rule f by an $m \times m$ matrix $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$, where $M = \{1, ..., m\}$. The rows indicate all the types of negotiator 1 and the columns are for the types of negotiator 2. We ignore, without loss of generality, types that deem no alternative acceptable from the matrix representation.

$$
f = \begin{array}{c|c|c|c|}
 & \theta_2^{x_1} & \cdots & \theta_2^{x_m} \\
\hline
\theta_1^{x_1} & f_{1,1} & \cdots & f_{1,m} \\
\hline
\vdots & \vdots & \ddots & \vdots \\
\hline
\theta_1^{x_m} & f_{m,1} & \cdots & f_{m,m} \\
\hline
\end{array}
$$

In this matrix, row (column) ℓ represents the preference of negotiator 1 (2) that finds all alternatives $\{x_k | k \leq \ell\}$ $\left(\{x_k | k \geq \ell\}\right)$ acceptable. Therefore, there is a unique mutually acceptable alternative in the main (first) diagonal of the matrix, i.e., $\{f_{\ell,\ell} \mid \ell \in M\}$. Note that there is no mutually acceptable alternatives for the type profiles that correspond to an entry in the upper half of the matrix.

Theorem 1. *Suppose that the regular extension map* Λ *satisfies logrolling. The mediation rule* f *is efficient, individually rational and strategy-proof if and only if the following hold:*

(i) *If* $\ell < j$, *then* $f_{\ell,j} = (o_X, y)$ *for some* $y \in Y$.

(ii) *If* $\ell = j$, *then* $f_{\ell,j} = (x_\ell, y_{m+1-\ell})$.

(iii) *(Adjacency) If* $\ell > j$, *then* $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ *and there exists a complete, transitive and strict precedence order* $\rhd$ *on* $\mathbf{B}$ *such that*

$$
f_{\ell,j} = \begin{cases} f_{\ell-1,j}, & \textit{if } f_{\ell-1,j} \rhd f_{\ell,j+1} \\ f_{\ell,j+1}, & \textit{oth.} \end{cases}
$$

[11] The proof is omitted as it directly follows from the arguments in our revelation principle result.

Example 2 (Adjacent rules): Let $X = \{x_1, x_2, x_3, x_4, x_5, o_X\}$, $Y = \{y_1, y_2, y_3, y_4, y_5, o_Y\}$, so the set of logrolling bundles is $\mathbf{B} = \big\{(x_1, y_5), (x_2, y_4), (x_3, y_3), (x_4, y_2), (x_5, y_1)\big\}$. A standard member of the family of adjacent rules is constructed by the following steps. It gives the outside option in issue X, bundled with some alternative from issue Y, whenever the negotiators have no mutually acceptable alternative in issue X. For our example, it is the bundle (o_X, y_3).

We fill the main diagonal with the members of the set of logrolling bundles, $\mathbf{B}$. In the first row and column, for example, we have (x_1, y_5). A reason for this is that the only mutually acceptable alternative is x_1 for the types in the first row and column. Therefore, deal-breakers property imply that an individually rational rule must suggest a bundle with x_1. Thus, we must have (x_1, y_5) in the first row and column because the adjacent rules always suggest a bundle from the set of logrolling bundles—a critical property of the adjacent rules that is necessary for strategy-proofness. For the rest of the matrix, i.e., the lower half of it, we need a strict precedence order over the logrolling bundles, $\mathbf{B}$. One example is

$$\rhd : (x_5, y_1) \;\rhd\; (x_1, y_5) \;\rhd\; (x_4, y_2) \;\rhd\; (x_2, y_4) \;\rhd\; (x_3, y_3)$$

Because the bundle (x_5, y_1) is ranked first, it beats all the other bundles in $\mathbf{B}$ in a binary comparison. Therefore, starting from the row and column of the bundle (x_5, y_1), all the rows below it and all the columns to the left of it should be filled with (x_5, y_1). Then the second bundle in the precedence order is (x_1, y_5), and it beats all the other bundles in $\mathbf{B}$ except (x_5, y_1). Thus, starting from the row and column of the bundle (x_1, y_5) on the main diagonal, all the empty rows below it and all the empty columns to the left of it should be filled with (x_5, y_1). Iterating this process for all the bundles in the precedence order will yield the following matrix:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_4}$	(x_1, y_5)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(o_X, y_3)
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

Figure 1:*A standard member of the adjacent rules family*

Note that there are various other complete binary relations over the bundles in $\mathbf{B}$. Theorem 1 proves that any complete, transitive and antisymmetric binary relations over

$\mathbf{B}$ generates a strategy-proof, efficient and individually rational mediation rule, and conversely any such mediation rule corresponds to a complete, transitive and antisymmetric binary relation over $\mathbf{B}$. In that sense, Theorem 1 characterizes the family of strategy-proof, efficient and individually rational rules.

Geometric Characterization of the Adjacent Rules

To provide further insight into the adjacent rules that are characterized in Theorem 1, and in particular, to gain a better understanding of the implications of part (*iii*) of Theorem 1, we provide a geometric characterization for these rules. This subsection assumes that the mediation rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$ satisfies the conditions (*i*) and (*ii*) of Theorem 1. For any $k \in M$ let b_k denote the logrolling bundle $(x_k, y_{m-k+1}) \in \mathbf{B}$.

Definition 6. *Consider $f_{k,k} = b_k$ and an entry that lies (weakly) to its southwest, $f_{\ell,j}$ with $j \leq k \leq \ell$. The **rectangle** induced by b_k and $f_{\ell,j}$, denoted by $\Re^{b_k}_{\ell,j}$, is the set of all entries of f in between rows k and ℓ and between columns k and j (inclusively). Namely, $\Re^{b_k}_{\ell,j} \equiv \bigcup_{\substack{j\leq s\leq k\\ k\leq t\leq \ell}} \{f_{t,s}\}$.*

Definition 7. *The **triangle** induced by an entry $f_{\ell,j}$, with $j \leq \ell$, denoted by $\triangle_{\ell,j}$, is the set of all entries of f in the triangular region that is bounded (inclusively) by the entry $f_{\ell,j}$, row ℓ, column j, and the main diagonal. Namely, $\triangle_{\ell,j} \equiv \bigcup_{j\leq t\leq \ell} \{f_{t,j}, f_{t,j+1}, ..., f_{t,t}\}$.*

Note that an entry on the main diagonal is a special triangle (and also a special rectangle) that consists of a singleton entry. Furthermore, the entire main diagonal of the mediation rule f and all the entries to its southwest constitute the largest possible triangle $\triangle_{m,1}$. Given a triangle $\triangle_{\ell,j}$, its entries that lie on the main diagonal are said to be on the *hypotenuse* of $\triangle_{\ell,j}$, and these entries are denoted by $\mathbf{B}_{\ell j} \equiv \{f_{j,j}, ..., f_{\ell,\ell}\} = \{b_j, ..., b_\ell\}$. A partition is called a *rectangular (triangular) partition* if and only if it is the union of disjoint rectangles (triangles).[12] To avoid confusion due to the slight abuse of terminology in the above definitions, it is worth underlining that a partition and any of its elements are collections of entries, i.e., indexes of rows and columns, of the matrix induced by rule f. We say $f_{\ell,j} = b$ when we need to specify that the mediation rule f is suggesting the bundle b at entry $f_{\ell,j}$. Finally, given a set $B \subseteq \mathbf{B}$ and a strict, transitive and complete precedence order $\rhd$ on $\mathbf{B}$, let $\underset{B}{\boldsymbol{max}} \rhd \equiv \{b \in B \mid b \rhd a \text{ for all } a \in B\backslash\{b\}\}$ denote the unique bundle in B with the highest precedence rank with respect to $\rhd$.

[12]Note that a rectangular partition consists of m disjoint rectangles. For example, $\{\Re^{b_k}_{k,1}\}^m_{k=1}$ and $\{\Re^{b_k}_{m,k}\}^m_{k=1}$ are two obvious rectangular partitions of $\triangle_{m,1}$. These two partitioning correspond respectively to what we will later refer to as the negotiator 1- and negotiator 2-optimal rules.

Theorem 2 (Geometric Characterization). *Consider a mediation rule f satisfying parts (i) and (ii) of Theorem 1. The following statements are equivalent:*

(1) *f satisfies part (iii) of Theorem 1.*

(2) *$\triangle_{m,1}$ has a rectangular partition such that f assigns a unique bundle to each rectangle in this partition.*[13]

(3) *Precedence order $\triangleright$ is such that $f_{\ell,j} = \underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}} \triangleright$.*

Part (2) of Theorem 2 states that an adjacent rule f can be represented as the union of m disjoint rectangular regions of the matrix induced by f, where each rectangle has a distinct corner entry on the main diagonal, containing the logrolling bundle that fills up the entire rectangle. Procedurally, these rectangles are obtained as follows: Given the precedence order, the highest ranked bundle on the hypotenuse of the largest triangle $\triangle_{m,1}$ fills up all the entries that are located to its southwest. This creates the first and the largest rectangle $\Re$, and generates a unique triangular partition of $\triangle_{m,1} \backslash \Re$. Next, pick any of these two triangles and let the highest ranked bundle on the hypotenuse of this triangle fill up all the entries that are located to its southwest. This leads to a second rectangle $\Re'$ as well as a unique triangular partition of $\triangle_{m,1} \backslash \{\Re, \Re'\}$. The process can be iterated this way until the entire triangle $\triangle_{m,1}$ is partitioned with disjoint rectangles.

Conversely, any such geometric set, namely any rectangular partition of $\triangle_{m,1}$, can be used to construct a precedence order and a corresponding adjacent rule. Part (3) of Theorem 2 states that for an efficient, individually rational and strategy-proof mediation rule f the entry $f_{\ell,j}$ is equivalent to the logrolling bundle with the highest precedence rank among the bundles that are on the hypotenuse of triangle $\triangle_{\ell,j}$. This eliminates the need to calculate immediately adjacent bundles of each entry as it was required by part (iii) of Theorem 1.

Special members of the adjacent rules family

There are some special members of the adjacent rules family. Negotiator 1 (2)-optimal adjacent rule, for example, is constructed by using the strict counterpart of the preference of negotiator 1 (2) over the logrolling bundles, $\mathbf{B}$, as the precedence order. For the same example above, the negotiator 1-optimal rule takes

$$\triangleright^1\!: (x_5, y_1) \;\triangleright^1\; (x_4, y_2) \;\triangleright^1\; (x_3, y_3) \;\triangleright^1\; (x_2, y_4) \;\triangleright^1\; (x_1, y_5)$$

[13] More formally, for any $\Re$ in the partition of $\triangle_{m,1}$ and any pair $a, b \in \Re$, $a = b$; but for any distinct pair $\Re, \Re'$ in the partition of $\triangle_{m,1}$, $a \in \Re$ and $b \in \Re'$ implies that $a \neq b$.

whereas the negotiator 2-optimal rule takes

$$\rhd^2\text{: } (x_1, y_5) \ \rhd^2 \ (x_2, y_4) \ \rhd^2 \ (x_3, y_3) \ \rhd^2 \ (x_4, y_2) \ \rhd^2 \ (x_5, y_1)$$

and they look as follow:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)				
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)			
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)		
$\theta_1^{x_4}$	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

Figure 2-a: *Negotiator 1-optimal rule*

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)				
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)			
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)		
$\theta_1^{x_4}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	
$\theta_1^{x_5}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Figure 2-b: *Negotiator 2-optimal rule*

Negotiator 1-optimal rule always picks negotiator 1's most preferred bundle among the *mutually acceptable logrolling bundles*. Although these rules are efficient, individually rational and strategy-proof, they are not impartial (favoring one negotiator over the another). There is another special member of the adjacent rules family that treats negotiators "symmetrically" whenever the mediation problem is symmetric.[14]

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)				
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)			
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)		
$\theta_1^{x_4}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(x_4, y_2)	
$\theta_1^{x_5}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Figure 3: *Compromise Rule*

Compromise rule and its characterization

[14]The mediation problem is symmetric if the number of alternatives in issues is an odd number. The problem is symmetric in this case because there is a unique median alternative in each issue, and thus, the number of alternatives (not including the outside option) better than the median alternative is the same for both negotiators. However, if there are two median alternatives, which is the case when the number of alternatives is even, then the mediation problem is not symmetric.

Although different indirect mechanisms would be used to describe the compromise rule, one of these indirect mechanisms is the following. In the **compromise (C)** rule the mediator picks one of the negotiators and asks him to provide a list of logrolling bundles with the restriction that the bundle (o_X, y_n) and the median bundle, (x_n, y_n) where n is the indices of the median alternative in each issue, must be in that list.[15] Then the mediator takes that list to the other negotiator and asks him to provide a shortlist from this list with the restriction that the bundle (o_X, y_n) must be in that shortlist. That is, the second negotiator is asked to eliminate at least one bundle from the list except the bundle (o_X, y_n). Then the mediator takes this shortlist to the first negotiator and asks him again to provide a shortlist of that shortlist with no constraint. The mediator goes back and forth between the negotiators until only one bundle remains.

Compromise rule is a special member of the adjacent rule family. It acts as though it is a negotiator 1 or 2-optimal rule if the median alternative in issue X is not mutually acceptable, and suggests the "median" bundle, (x_n, y_n), otherwise. In addition to being efficient, individually rational and strategy-proof, compromise rule minimizes rank variance within the class of efficient, individually rational and strategy-proof rules. We prove this point next.

Given the negotiators' fixed preferences over alternatives (not including the outside option), let $r_i(z) \in M$ denote negotiator i's ranking of the alternative $z \in Z \in \{X, Y\}$.[16] Given a mediation rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$, let $f_{\ell,j} = (f^X_{\ell j}, f^Y_{\ell j}) \in X \times Y$ denote the bundle it proposes when the negotiators' types are $\theta_1^{x_\ell}$ and $\theta_2^{x_j}$. Therefore, the *rank variance* of the bundle $f_{\ell,j}$ is defined by[17]

$$var(f_{\ell,j}) \equiv \sum_{i\in I} \left(r_i(f^X_{\ell j})\right)^2 + \left(r_i(f^Y_{\ell j})\right)^2.$$

Thus, the rank variance of the mediation rule f is

$$Var(f) = \sum_{\ell=1}^{m}\sum_{j=1}^{m} var(f_{\ell,j}).$$

A bundle including a/the median alternative in both issues has the smallest rank variance and bundles (x_1, y_1) and (x_m, y_m) have the highest rank variance. Intuitively, rank variance of a bundle is a measure of the extent to which that bundle favors one

[15] More formally, $n \in \{\overline{n}, \underline{n}\}$, where $\overline{n} = \lceil \frac{m+1}{2} \rceil$ and $\underline{n} = \lfloor \frac{m+1}{2} \rfloor$. If m is odd, then there is a unique median alternative in each issue because $\overline{n} = \underline{n} = \frac{m+1}{2}$. If there are two median alternatives, namely m is even, then the mediator picks any one of them.

[16] We ignore the outside option from the rank calculations without loss of generality because we will restrict our attention to individually rational and efficient rules.

[17] One may assign different weights to the issues in the definition of rank variance. The results still go through without any loss.

negotiator over the other negotiator. In this sense, the higher the rank variance of a bundle or a mediation rule, the more biased its treatment is. Alternately, the lower the rank variance of a mediation rule, the more impartial it is. Normatively speaking, a rule that aims to minimize rank variance can be viewed as one choosing "the center of gravity" or the "middle ground" along the tradeoffs the negotiators are facing.

Definition 8. *For any $k \in M$, let the bundle $b_k = (x_k, y_{m-k+1})$ be the logrolling bundle in $\mathbf{B}$. A rule is a compromise rule, denoted $f^C = [f_{\ell,j}]_{(\ell,j)\in M^2}$, if it is an adjacent rule (as described in Theorem 1) that is associated with a precedence order $\rhd^C$, where $b_n \rhd^C b_{n-1} \rhd^C \ldots \rhd^C b_1$ and $b_n \rhd^C b_{n+1} \rhd^C \ldots \rhd^C b_m$ with n being the indices of the median alternative in both issues, and $f^C_{\ell,j} = (o_X, y_n)$ whenever $\ell < j$.*

Note that there is a unique compromise rule if m is odd. If m is even, however, a compromise rule prescribes one of four types of outcomes.[18]

Theorem 3. *A mediation rule minimizes rank variance within the class of efficient, individually rational and strategy-proof rules if and only if it is a compromise rule.*

5. A Necessary and Sufficient Condition for Strategy-Proofness

Definition 9. *The extension map Λ is **consistent** if it is regular and if for all $i \in N$, $\theta_i \in \Theta_i$, and $b = (x, y), b' = (x', y') \in X \times Y$ with $x, x' \in A(\theta_i)$ we have*

1. $\Big[b\ I_i\ b' \text{ for all } R_i \in \Lambda(\theta_i)\Big] \Longrightarrow \Big[b = b'\Big]$, *and*

2. $\Big[b\ R_i\ b' \text{ for all } R_i \in \Lambda(\theta_i)\Big] \Longrightarrow \Big[b\ R_i\ b' \text{ for all } R_i \in \Lambda(\theta'_i) \text{ and } \theta'_i \in \Theta_i\Big]$.[19]

The first condition implies that if negotiators are indifferent between two bundles at all admissible preferences, then these two bundles must be the same. The second condition imposes that if some types of a negotiator can unambiguously rank two acceptable bundles, then all types of that agent, who deem these two bundles acceptable, should unambiguously and similarly rank them.

Definition 10. *Let B be a nonempty subset of $\mathbf{B}$ and the bundles $b = (x_\ell, y_{m-\ell+1}), b' = (x_j, y_{m-j+1})$ are in B. Then b' is **adjacent to bundle** b **in** B if there exists no bundle $(x_k, y_{m-k+1}) \in B$ with $\ell < k < j$ (or $j < k < \ell$) whenever $\ell < j$ (or $j < \ell$). We call such two bundles adjacent in B and denote $b' \in B(b)$.*

[18] It depends on whether $b_{\overline{n}}$ or $b_{\underline{n}}$ has the highest precedence order and whether $y_{\overline{n}}$ or $y_{\underline{n}}$ is chosen when no mutually acceptable alternative in issue X exists.

[19] Let I_i denote the indifference part of R_i, i.e., $b\ I_i\ b'$ if and only if $b\ R_i\ b'$ and $b'\ R_i\ b$.

Note that adjacency is symmetric, that is if bundle b is adjacent to b' in B, then b' is also adjacent to b in B.

Definition 11. *A binary relation $\rhd$ on $\mathbf{B}$ is* ***complete with respect to adjacency on*** *$B \subseteq \mathbf{B}$ if for any two distinct and adjacent bundles b, b' in B we have either $b \rhd b'$ or $b' \rhd b$.*

Let $\rhd$ be a binary relation over the set of logrolling bundles. Set $B^0_\rhd = \mathbf{B}$ and recursively define $B^k_\rhd = \{b \in B^{k-1}_\rhd \mid b \rhd b' \text{ for some } b' \in B^{k-1}_\rhd(b)\}$ for $k = 1, 2, \ldots$. If the binary relation $\rhd$ is antisymmetric, then the number of elements in each $B^k_\rhd$ is at most $m - k$ for $k < m$ and 1 for all $k \geq m$.

Definition 12. *The binary relation $\rhd$ on $\mathbf{B}$ is called* ***connected*** *if it is complete with respect to adjacency on all $B^k_\rhd$, $k = 0, 1, \ldots$.*

Definition 13. *The extension map Λ admits a binary relation $\rhd$ on $\mathbf{B}$ that* ***concatenates negotiators' preferences*** *if for any two distinct logrolling bundles $b = (x_\ell, y_{m-\ell+1}), b' = (x_j, y_{m-j+1}) \in \mathbf{B}$, $b \rhd b'$ implies*

1. *$b\ R_1\ b'$ for all $R_1 \in \Lambda(\theta_1)$ and all $\theta_1 \in \Theta_1$ with $x_\ell, x_j \in A(\theta_1)$ whenever $j < \ell$,*

2. *$b\ R_2\ b'$ for all $R_2 \in \Lambda(\theta_2)$ and all $\theta_2 \in \Theta_2$ with $x_\ell, x_j \in A(\theta_2)$ whenever $j > \ell$.*

Definition 14. *The extension map Λ satisfies* ***weak logrolling*** *if it admits a transitive, antisymmetric and connected binary relation $\rhd$ over the set of logrolling bundles that concatenates negotiators preferences. In that instance we call $\rhd$* ***admissible with respect to*** *Λ.*

Denote a mediation rule by $f^\rhd$ if the mediation rule and the precedence order $\rhd$ satisfy the conditions $(i) - (iii)$ of Theorem 1.

Theorem 4. *Suppose that Λ is consistent. The mediation rule f is efficient, individually rational and strategy-proof if and only if Λ satisfies weak logrolling and $f = f^\rhd$ for some $\rhd$ that is admissible with respect to Λ.*

6. Role of the Assumptions

As we argued earlier two issue is without loss of generality. If there are more than 2 issues in the dispute problem, then we can regroup these issues as those that fall under the category of issue X and category Y. Under this re-grouping, each alternative would now be a vector of alternatives, one for each issue. The negotiators' preferences over these vectors of alternatives need not be diametrically opposed. However, in light of Proposition 1, two issues with diametrically opposed preferences is without loss of generality as long as the negotiators' preferences are monotonic.

(Weak) Logrolling is a critical assumption for the possibility results. It is a well-defined concept whenever the number of alternatives in issue Y is greater than the number of alternatives in issue X, i.e., $\#Y \geq \#X$. In case $\#Y > \#X$ there may exist many one-to-one function that permits reversal of preferences. Any such function suffices to generate a class of individually rational, efficient and strategy-proof mediation rules that we characterize in Theorem 1.

Include followings: Example showing that deal-breakers property is not necessary for possibility. Issue-wise voting doesn't work. More than two negotiators.

Symmetric Treatment of the Outside Options

In this section, we relax the assumption that $y\ \theta_i^Y o_Y$ for all $i \in \mathrm{I}$ and $y \in Y \setminus \{o_Y\}$. Instead, the negotiators' ranking of the outside option, o_Y, is their private information, as is the case for issue X. Thus, $\Theta_i = \Theta_i^X \times \Theta_i^Y$ denotes the set of all **types** of negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ is the set of all type profiles. We also relax our assumption of regularity for the negotiators' ranking over the bundles, and suppose that they satisfy monotonicity, i.e., condition (i) of Definition 1, and the following modification of condition (ii), i.e., deal-breakers. We need a modified version of this assumption because both issues X and Y have unacceptable alternatives now.

Definition 15. *Under the symmetric treatment of the outside options, the extension map Λ is regular if the followings hold for all i, $\theta_i \in \Theta_i$ and all $R_i \in \Lambda(\theta_i)$:*

i. [Monotonicity] For any $x, x' \in X$ and $y, y' \in Y$ with $(x, y) \neq (x', y')$,

$$(x, y)\ P_i\ (x', y') \text{ whenever } \left[x\ \theta_i^X\ x' \text{ or } x = x'\right] \text{ and } \left[y\ \theta_i^Y\ y' \text{ or } y = y'\right].$$

ii. [DB] $(o_X, o_Y)\ P_i\ (x, y)$ whenever $o_X\ \theta_i^X\ x$ or $o_Y\ \theta_i^Y\ y$.

Proposition 2. *Under the symmetric treatment of the outside options, there is no mediation rule f that is strategy-proof, individually rational and efficient.*

Note that this impossibility result can easily be carried out to a single issue or more than two issue contexts. A rule that always picks the pair (o_X, o_Y) is strategy-proof but not efficient. A dictatorship is efficient and strategy-proof but not individually rational.

7. Mediation with Continuum of Alternatives

Suppose now that the issues X and Y are two closed and convex intervals of the real line. The outside options, o_X and o_Y, may or may not be the elements of these sets. We assume, without loss of generality, that $X = Y = [0, 1]$, with the interpretation that the negotiators aim to divide a unit surplus in each issue. In order to keep the notation consistent with the previous section, let a bundle $b = (x, y)$ indicate that negotiator 2 gets x and y in issues X and Y, respectively, and thus, negotiator 1 gets $1 - x$ and $1 - y$, respectively. Namely, each alternative in each issue indicates what negotiator 2 receives. Agents having diametrically opposing preferences on each issue means that for any issue $Z \in \{X, Y\}$ and two alternatives $z, z' \in Z$, negotiator 1 (2) prefers z to z' whenever $z < z'$ ($z > z'$). The value/ranking of the outside option o_X in issue X is each negotiators' private information. However, the value/ranking of the outside option o_Y in issue Y is common knowledge, and both negotiators prefer all $y \in Y$ to o_Y.

For any $\ell \in [0, 1]$, type ℓ of negotiator 1 (2), denoted by θ_1^ℓ (θ_2^ℓ), prefers the outside option o_X to all alternatives $k \in [0, 1]$ with $\ell < k$ ($\ell > k$).[20] Parallel to the discrete case, we denote the mediation rule $f = [f_{\ell,j}]_{(\ell,j)\in[0,1]^2}$ where $f_{\ell,j} = f(\theta_1^\ell, \theta_2^j)$ for all $0 \leq \ell, j \leq 1$.[21] The negotiators have no mutually acceptable alternative in issue X at type profile $(\theta_1^\ell, \theta_2^j)$ when $\ell < j$. The set of mutually acceptable alternatives is $A(\theta_1^\ell, \theta_2^j) = [j, \ell]$ whenever $\ell \geq j$. We use Θ_i as the set of all types of negotiator i and $\theta_i \in \Theta_i$ as the generic element whenever we do not need to specify the type's least acceptable alternative. Monotonicity and deal breakers assumptions of the regularity condition in the previous section directly applies here. The same is true for the definition of strategy-proofness, individual rationality and efficiency. We need a slight modification in the logrolling property for the next characterization result akin to Theorem 1.

Definition 16. *The extension map Λ satisfies logrolling if there exists a unique function $t : X \to [0, 1]$ such that for all i, $\theta_i \in \Theta_i$, $R_i \in \Lambda(\theta_i)$ and all $x, x' \in A(\theta_i)$ with $x \ \theta_i \ x'$, we have $\big(x', t(x')\big) \ R_i \ \big(x, t(x)\big)$.*

[20] Therefore, all k with $\ell \geq k$ ($\ell \leq k$) are deemed acceptable by type θ_1^ℓ (θ_2^ℓ) of negotiator 1 (2).

[21] We assume, without loss of generality, that each negotiator has at least one acceptable alternative. Therefore, there is no type profile where a negotiator deems all alternatives unacceptable.

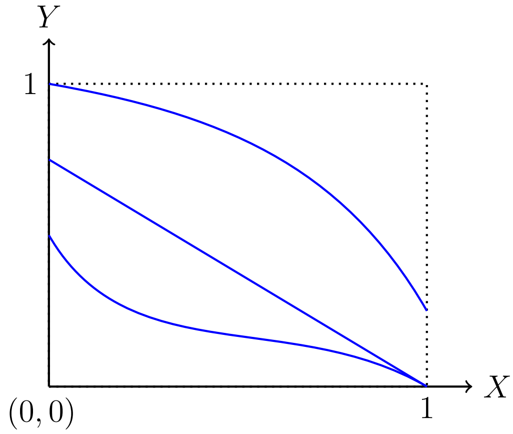


Figure 4: Possible Set of Logrolling Bundles

Each t function in Figure 4 (in fact any such decreasing function) would be used to represent the set of logrolling bundles. Uniqueness of the function t ensures unique set of logrolling bundles. Uniqueness is not necessary for a possibility result. In fact, if there were multiple t's satisfying logrolling property, then each one would generate a different set of logrolling bundles and a separate family of individually rational, efficient and strategy-proof rules of the form that we characterize in the next result.[22] Therefore, the set of logrolling bundles is $\mathbf{B}$, and so, for all values of $\ell, j \in [0,1]$ with $j \leq \ell$, $\mathbf{B}_{\ell j} = \{(x,y) \in \mathbf{B} \mid j \leq x \leq \ell\}$ denotes the set of all mutually acceptable logrolling bundles at type profile $(\theta_1^\ell, \theta_2^j)$.

Define $\rhd$ to be a complete, transitive and antisymmetric binary relation over the set of logrolling bundles. When $(\mathbf{B}, d)$ is a metric space with a proper metric d, $\mathbf{B}_{\ell j}$ with $\ell \geq j$ is a non-empty and compact subset of the set of logrolling bundles.

Definition 17. *The binary relation $\rhd$ is said to be* ***quasi upper-semicontinuous over*** $\mathbf{B}_{\ell j}$ *with $\ell \geq j$ if for all $a, c \in \mathbf{B}_{\ell j}$ with $a \neq c$, $a \rhd c$ implies that there exists a bundle $a' \in \mathbf{B}_{\ell j}$ and a neighborhood $\mathcal{N}(c)$ of c such that $a' \rhd b$ for all $b \in \mathcal{N}(c) \cap \mathbf{B}_{\ell j}$.*[23]

Therefore, the binary relation $\rhd$ is quasi upper-semicontinuous if it is quasi upper-semicontinuous over all compact subsets $\mathbf{B}_{\ell j}$ of $\mathbf{B}$. A bundle $b^\star \in \mathbf{B}_{\ell j}$ is said to be a maximal element of the binary relation $\rhd$ on $\mathbf{B}_{\ell j}$, i.e., $b^\star \in \underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}}$ if $b^\star \rhd b$ for all $b \in \mathbf{B}_{\ell j}$. Theorem 1 in Tian and Zhoub (1995) proves that quasi upper-semicontinuity is both necessary and sufficient for $\rhd$ to attain its maximum on all compact subsets $\mathbf{B}_{\ell j}$ of $\mathbf{B}$. Therefore, the analogous version of Theorem 1 in the continuous case reads as follows.

Theorem 5. *Suppose that the regular extension map Λ satisfies logrolling. The mediation rule f is efficient, individually rational and strategy-proof if and only if there exists a*

[22]However, if multiple t functions were present, then the domain would admit other efficient, individually rational and strategy-proof rules that are not covered by Theorem 5.

[23]This is Definition 2 in Tian and Zhoub (1995).

complete, transitive, antisymmetric, and quasi upper-semicontinuous binary relation $\rhd$ *over the set of logrolling bundles* $\mathbf{B}$ *and* $y \in Y \setminus \{o_Y\}$ *such that*

$$f_{\ell,j} = \begin{cases} (o_X, y), & \text{if } \ell < j, \\ \max\limits_{\mathbf{B}_{\ell j}} \rhd, & \text{oth.} \end{cases}$$

Analogous to the discrete case, we use the following continuously indexed matrix to describe a mediation rule f.

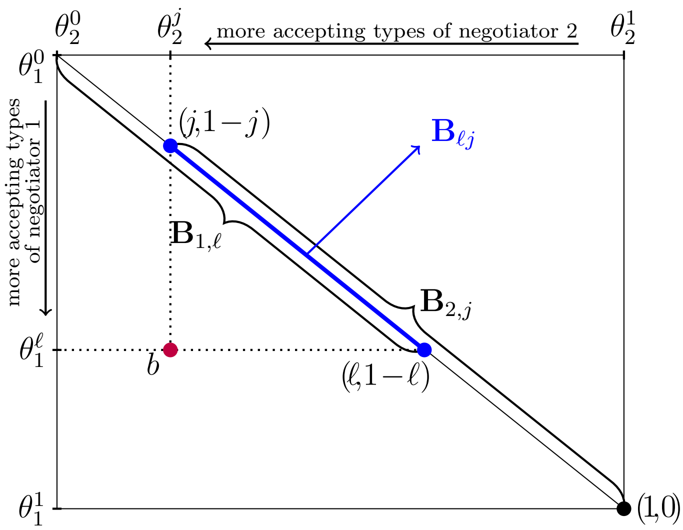


Figure 5: Adjacent rules in the continuous case

The rows, i.e., the vertical axis, correspond to the types of negotiator 1 and columns, i.e., the horizontal axis, indicate all possible types of negotiator 2. Each point on the main diagonal represents a logrolling bundle for the mediation rule that is described in Theorem 5, and each logrolling bundle appears only once on this diagonal. The bundle b, for example, represents the value of f when the true type of negotiator 1 and 2 are θ_1^ℓ and θ_2^j, respectively. When the true type profile is (θ_1^1, θ_2^1), negotiator 1 finds all alternatives acceptable and negotiator 2 deems all alternatives except 1 unacceptable. We assume, without loss of generality, $t(x) = 1 - x$ for the rest of the paper. Namely, the set of logrolling bundles is $\mathbf{B} = \{(x, y) \in [0,1]^2 \mid y = 1 - x\}$. Then the only mutually acceptable logrolling bundle is $(1, 0)$ at type profile (θ_1^1, θ_2^1).

The set of all acceptable logrolling bundles for type θ_1^ℓ of negotiator 1 is denoted by $\mathbf{B}_{1,\ell}$, which consists of all the logrolling bundles on the upper portion of the main diagonal, starting from the north west corner bundle, $(0, 1)$, and goes all the way down to the bundle $(\ell, 1 - \ell)$. That is, $\mathbf{B}_{1,\ell} = \{(k, 1 - k) \in \mathbf{B} \mid 0 \leq k \leq \ell\}$. Similarly, the set of all acceptable logrolling bundles for type θ_2^j of negotiator 2 is represented by $\mathbf{B}_{2,j}$ and it consists of all the bundles on the lower portion of the main diagonal, i.e., all bundles from $(j, 1-j)$ to $(1, 0)$. Namely, $\mathbf{B}_{2,j} = \{(k, 1-k) \in \mathbf{B} \mid j \le k \le 1\}$. Thus, the set of all mutually acceptable logrolling bundles at the type profile $(\theta_1^\ell, \theta_2^j)$ is the intersection of these two sets, i.e., $\mathbf{B}_{\ell j} = \mathbf{B}_{1,\ell} \cap \mathbf{B}_{2,j}$. Theorem 5 states that bundle b is the logrolling bundle that maximizes $\rhd$ within the set $\mathbf{B}_{\ell j}$ (see Figure 5). A maximal bundle uniquely exists because $\rhd$ is antisymmetric.

Preferences that Satisfy Logrolling

One of the critical requirements for our results is logrolling. Simply put, it requires that attractiveness of the alternatives in issue Y are sufficiently dispersed so that negotiators are willing to trade one acceptable alternative in issue X with another acceptable alternative in X. To see this point, consider the following simple example:

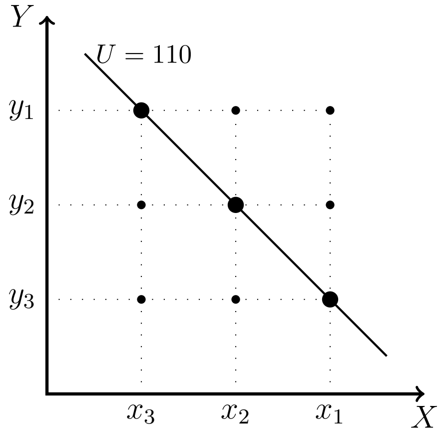
Example 3: Suppose that preferences over the bundles are additively separable and each issue has three alternatives. Let $u(.)$ and $v(.)$ represent preferences over issues X and Y, respectively. Therefore, $U(x, y) = u(x) + v(y)$ is the utility function over the bundles.[24]

X	$u(.)$	Y	$v(.)$	$X \times Y$	$U(.)$
x_1	100	y_1	20	(x_3, y_1)	110
x_2	98	y_2	12	(x_2, y_2)	110
x_3	90	y_3	10	(x_1, y_3)	110

The utility functions (preferences) in this simple example satisfy logrolling although the worst alternative in issue X is 4.5 times more valuable, in absolute terms, than the most valuable alternative in issue Y.

In standard consumer theory, we represent preferences over bundles by drawing corresponding indifference curves on commodity space, where each axis corresponds the quantity of a particular commodity. In the current model, issues serve the same role with commodities. However, distance between two alternatives is irrelevant in our setup as we abstract away from quantities. In our discrete setup, marginal rate of substitution would be the rate at which a negotiator can give up some *number of alternatives* in one issue in exchange for the other issue while maintaining the same level of utility. Therefore, without loss of generality, we can place all alternatives equidistantly. Also, we place less preferred alternatives closer to the origin, implying (together with monotonicity) higher indifferent curves as we move northeastern direction. For the previous numerical example, therefore, preferences of the type that deems all alternatives acceptable can be pictured as follows:

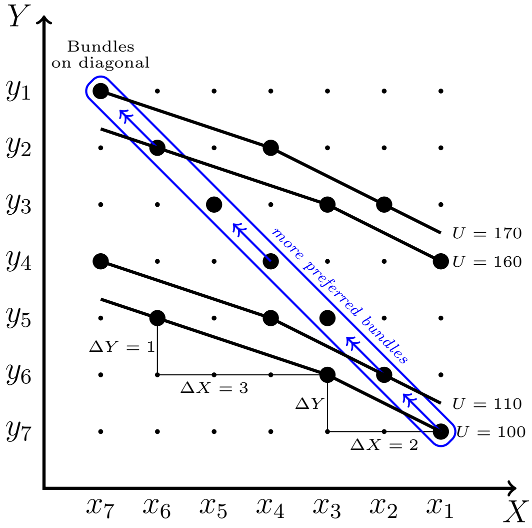
[24] For completeness, one may assume that all types get very large disutility from unacceptable alternatives, including the outside option.

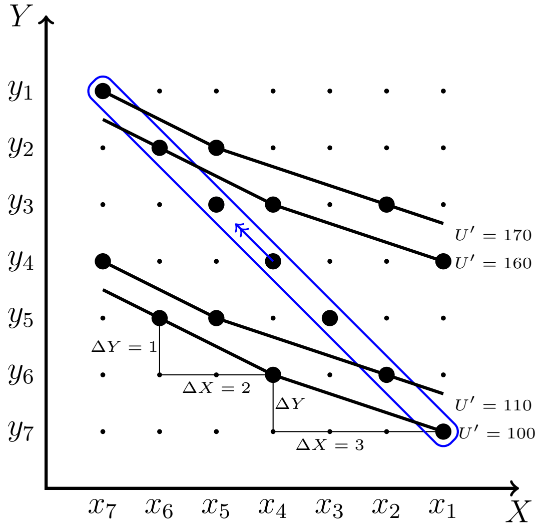


As it is evident from this graph, logrolling is a property of the bundles that are placed on the *diagonal*. Logrolling requires that these bundles are either lying on the same indifference curve or on higher indifference curves as we move along the diagonal in northwestern direction (and southeast direction for the other negotiator). The marginal rate of substitution at diagonal bundles is one for our example because it requires one alternative to trade between the issues to keep the negotiator's utility the same. The utility function we picked behaves *as if* issues are perfect substitutes. But it is hardly possible to make a concrete statement with only nine bundles. On the other hand, logrolling is consistent with all range of utility functions that have "convex" or "concave" indifference curves. Consider the following two utility functions, U and U':

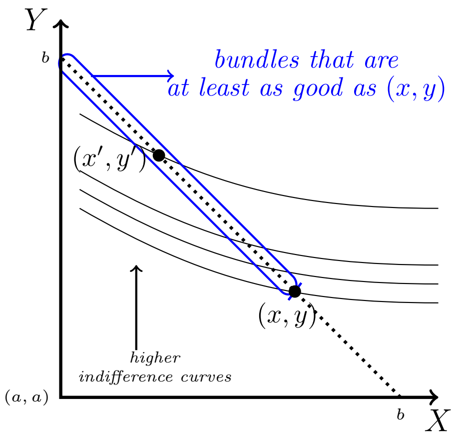
X	$u(.)$	$u'(.)$	Y	$v(.)$	$X \times Y$	$U(.) = u(.) + v(.)$	$U'(.) = u'(.) + v(.)$
x_1	100	100	y_1	120	(x_7, y_1)	170	170
x_2	90	90	y_2	100	(x_6, y_2)	160	160
x_3	80	85	y_3	80	(x_5, y_3)	145	150
x_4	70	80	y_4	60	(x_4, y_4)	130	140
x_5	65	70	y_5	40	(x_3, y_5)	120	125
x_6	60	60	y_6	20	(x_2, y_6)	110	110
x_7	50	50	y_7	0	(x_1, y_7)	100	100

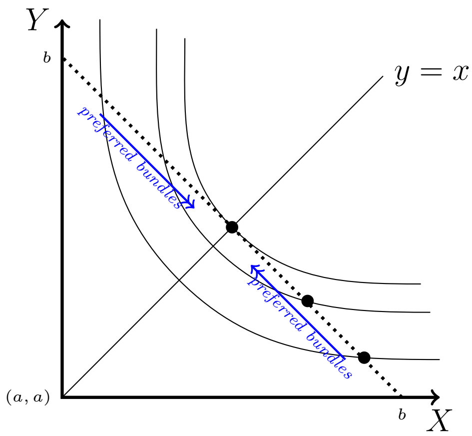
Both utility functions satisfy logrolling and their indifference curves are drawn in the following two graphs. As it is also clear from these graphs, the marginal rate of substitution of the utility function U (U') is increasing (decreasing) as we move to the right in the X axis, which are interpreted as indifference curves for U (U') being concave (convex).





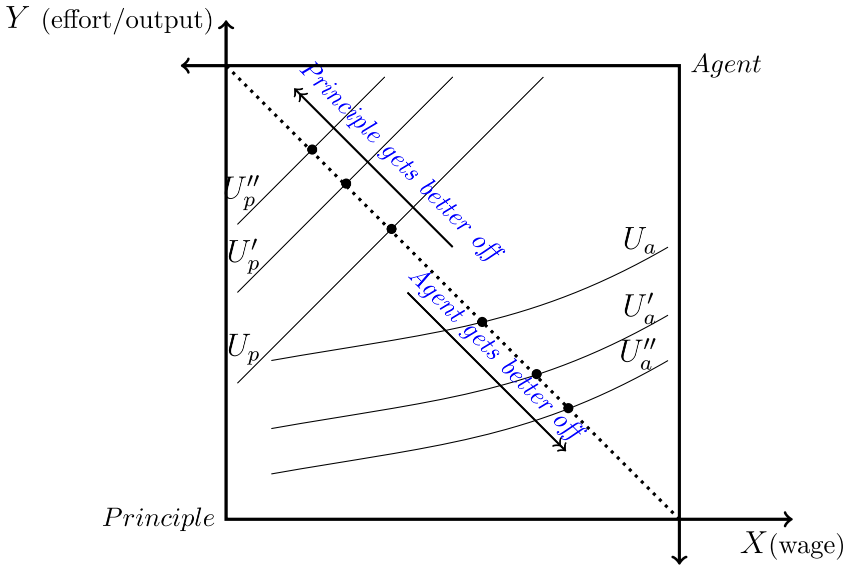
In a more standard setup where alternatives for issues X and Y represent quantities of two commodities, ranging over some interval $[a, b] \subseteq \mathbb{R}^2$, logrolling will be satisfied for all utility functions with marginal rate of substitution (MRS) that is less than or equal to one at all points of the diagonal. For example, a utility function $U(x, y) = \sqrt{x} + y$ does satisfy this condition whenever $1/4 \leq a$. This is true because upper counterset of a bundle (x, y) that is on diagonal includes all the other bundles (x', y') on the diagonal that are situated northwest of the original bundle (x, y).





In fact, all utility functions with convex indifference curves with $MRS_{x=y} \leq 1$ satisfy a weaker condition of logrolling, where logrolling holds only for the first half of the alternatives in issue X. This weaker condition is sufficient to guarantee strategy-proof rules. An example for these utility functions would be $U(x, y) = x^\alpha y^\beta$ where $\alpha/\beta \leq 1$. The second graph above demonstrates why such convex utility functions satisfy this weaker condition.

Alternatively, one may consider a moral hazard situation between a principle and an negotiator, where X denotes the domain for wage and Y denotes different levels of effort/output. Consistent to this framework, let the agent's and the principle's utility functions are $U_a(x, y) = u_a(x) - v_a(y)$ and $U_p(x, y) = v_p(y) - u_p(x)$, respectively, where all u's and v's are increasing functions. Along the diagonal, the principle's indifference curves increase as we move in northwestern direction (and the worker's indifference curves increase as we move in southeastern direction) as required by the logrolling condition. For a simple example, one may consider $U_w(x, y) = x - y^2$ and $U_p = y - x$, which we depict below.



Relating to the impossibility result of Myerson and Satterthwaite (83)

This section explores the underlying factors that are potentially absent in Myerson and Satterthwaite (83) [MS] model, which may lead to the possibility result in our case. Drawing a one-to-one relationship between MS and the current model would be pushing the limits of these two models too much as there are notable differences between the two: One of them is that MS takes agents' preferences over bundles as a primitive of the model, whereas we start with preferences over alternatives for each issue, i.e., the marginal preferences, and then generate the set of acceptable preferences over bundles from these. Nonetheless, this section aims to shed some light on the similarities and differences between these two models and discuss whether these differences cause the stark difference in conclusions.

It is already well-known by the literature that denseness of the type space is one of the reasons for the impossibility result in MS. However, this is not the driving force for our strategy-proofness result as we formally show it in the continuous mediation case. Our assumptions over the individual preferences, i.e., diametrically opposing preferences over alternatives, monotonicity, deal-breaker or logrolling, are other potential suspects. However, as we discuss briefly below, the quasi linear preferences that are used in MS do satisfy all these assumptions. One important difference between these two models is that

MS treats both issues symmetrically, and so MS is effectively a single-issue, not a multi-issue, negotiation problem. This difference seems to be the main driving force behind our possibility result.

MS considers a bilateral trade between a seller, who owns an indivisible good, and a buyer, who likes to buy this good, as a mechanism design problem. The mechanism (p, x) has two components; the probability of trade, p, and the transfer, x, both of which are functions of the players' reports. If no trade occurs, then $x = p = 0$ (the outside option), and so both players receive zero utility. The utility functions are $U_b = v_b p - x$ for the buyer and $U_s = x - v_s p$ where the valuations v_b, v_s are the players' private information. Consider for simplicity that both players' valuations are distributed over the unit interval $[0, 1]$ according to some probability distribution.

One may map this setup to our two-issue framework, with a continuum of types, where the first issue is the probability of trade, i.e., p, and the second issue is the amount of transfer, i.e., x. It is clear from the utility functions that agents preferences over the individual issues are diametrically opposed. That is, for any fixed value of x, the buyer gets better off as p decreases from 1 to 0 and the seller gets worse off as p decreases from 1 to 0. Similarly, for any fixed value of p, the buyer gets better off as x decreases from 1 to 0 and the seller gets worse off as x decreases from 1 to 0. It is relatively easy to verify that these quasi-linear preferences over bundles satisfy monotonicity and logrolling. The deal-breaker property holds in the MS setup as well: For example, for type $v_b = 0.5$ of the buyer, there is no $p \in [0, 1]$ that makes transfers $x > 0.5$ acceptable as these transfers induce negative utilities for the buyer.

In our setup, each issue has separate outside option whereas MS assumes joint outside option for the issues, i.e., no trade. However, this difference does not seem to play any significant role. More importantly, MS corresponds to our symmetric two issue case. Although the utility of the outside option in MS in each issue is 0, the ranking of the outside option is the negotiators' private information. This is true because for the buyer, for example, the set of acceptable alternatives in issue p must satisfy $p \geq \frac{x}{v_b}$ for any fixed value of x, and this set is the buyer's private information as v_b is not common knowledge. Therefore, the set of acceptable outcomes (or the ranking of the outside option in each individual issue) are the players' private information, as it is the case in our symmetric case. We show, in the symmetric treatment of the outside option, that there is no individually rational and ex-post efficient strategy-proof mediation rules. We prove this point next.

Appendix

Proof of Proposition 1: Let $\widetilde{A} \subseteq A$ be the set of alternatives that survive the elimination of Pareto inefficient alternatives. Namely, none of the alternatives in $\widetilde{A}$ is Pareto inefficient. Re-number the elements in $\widetilde{A}$, and so suppose, without loss of generality, that $\widetilde{A} = \{x_1, ..., x_m\}$ where $m \geq 2$, and negotiator 1 ranks alternatives as $x_k \; \widetilde{\theta}_1 \; x_{k+1}$. If x_m is not the best alternative for $\widetilde{\theta}_2$ on $\widetilde{A}$, then there must exist some x_k where $k < m$ such that $x_k \; \widetilde{\theta}_2 \; x_m$. But this contradicts with the assumption that x_m is not Pareto inefficient. Thus, negotiator 2 must rank x_m as the top alternative. With a similar reasoning, if x_{m-1} is not negotiator 2's second best alternative, then it must be Pareto inefficient, contradicting with the assumption that x_{m-1} survives after deletion of Pareto inefficient alternatives. Iterating this logic implies that the rankings of the negotiators must be diametrically opposed.

Proof of Theorem 1: Note that if Λ is regular and satisfies logrolling, then Theorem 1 is a corollary to Theorem 4. This is true because, in this case, Λ satisfies weak logrolling and all complete, transitive and antisymmetric binary relations over the set of logrolling bundles, $\mathbf{B}$, are admissible with respect to Λ. Thus, the mediation rule f is efficient, individually rational and strategy-proof if and only if f satisfies the conditions $(i) - (iii)$ in the statement of Theorem 1.

Proof of Theorem 2:

We start with (1) $\Rightarrow$ (2). Suppose f satisfies parts (ii) and (iii) of Theorem 1. The first diagonal contains all the bundles in $\mathbf{B}$, which is the hypotenuse of $\triangle_{m,1}$. First, consider the highest ranked logrolling bundle on the hypotenuse of $\triangle_{m,1}$ and let $f_{r_1,r_1} = b_{r_1} = \boldsymbol{\underset{\mathbf{B}}{max}} \; \rhd$ where $1 \leq r_1 \leq m$. Iteratively applying the adjacency requirement, starting from the hypotenuse, implies that all the entries on row r_1 to the left of entry f_{r_1,r_1}, all the entries on column r_1 below entry f_{r_1,r_1} and all the entries in between must fill up with bundle b_{r_1} because b_{r_1} has the highest rank. Thus, the rectangle $\Re^{b_{r_1}}_{m,1}$ fills up with b_{r_1}. Let $\Re^{b_{r_1}}_{m,1}$ be the first element of the rectangular partition of $\triangle_{m,1}$. Note that, when $m \geq 3$, the so-far-unfilled $\triangle_{m,1} \backslash \Re^{b_{r_1}}_{m,1}$ consists of at least one triangle (if $r_1 \in \{1, m\}$) and at most two triangles (if $r_1 \notin \{1, m\}$).

Next, take an arbitrary triangle $\triangle_{a,b} \in \triangle_{m,1} \backslash \Re^{b_{r_1}}_{m,1}$. Note that either $a = r_1$ and $b = 1$, or $a = m$ and $b = r_1 + 1$. Let $f_{r_2,r_2} = b_{r_2}$, where $r_2 \neq r_1$, denote the highest ranked logrolling bundle on the hypotenuse of $\triangle_{a,b}$. Then iteratively applying the adjacency requirement, starting from the hypotenuse of $\triangle_{a,b}$, implies that all the so-far-unfilled entries on row r_2 to the left of entry f_{r_2,r_2}, all the so-far-unfilled entries on column r_2 below entry f_{r_2,r_2} and all entires in between must fill up with bundle b_{r_2} because b_{r_2} has the highest rank among the bundles on the hypotenuse of $\triangle_{a,b}$. Thus, let $\Re^{b_{r_2}}_{a,b}$ denote the second element of the rectangular partition of $\triangle_{m,1}$.

Note that the so-far-unfilled set $\triangle_{m,1}\backslash\left\{\Re^{b_{r_1}}_{m,1}\cup\Re^{b_{r_2}}_{a,b}\right\}$ consists of at least one triangle. Iterate this reasoning and at each step pick a triangle from the so-far-unfilled subset of $\triangle_{m,1}$ and fill its corresponding rectangle with the highest ranked bundle on its hypotenuse. By the finiteness of the problem, the rectangular partition is obtained in m steps.

We next show (2) $\Rightarrow$ (1). Consider a rectangular partition $\mathcal{P}^1$ of $\triangle_{m,1}(\equiv\triangle^1)$ such that for any $\Re\in\mathcal{P}^1$, $a,b\in\Re$ implies $a=b$. Let $\Re^{b_{r_1}}\subset\triangle^1$ be the rectangle that includes the entry at the bottom left corner of triangle $\triangle^1$, i.e., $f_{m,1}$. We construct a precedence order $\rhd$ as follows. Let b_{r_1} have a higher precedence rank than any other bundle on the hypotenuse of $\triangle^1$. Namely, let $b_{r_1}\rhd b$ for all $b\in\mathbf{B}\backslash\{b_{r_1}\}$. Next consider $\triangle^1\backslash\Re^{b_{r_1}}$ which has a triangular partition $\mathcal{P}^2$ that consists of at most two triangles.

Take an arbitrary triangle $\triangle^2\in\mathcal{P}^2$ and let $\Re^{b_{r_2}}\subset\triangle^2$ denote the rectangle that includes the entry at the bottom left corner of triangle $\triangle^2$. Then let b_{r_2} have a higher precedence rank than any other bundle on the hypotenuse of $\triangle^2$, i.e., if $r_2>r_1$, then let $b_{r_2}\rhd b$ for all $b\in\{b_1,...,b_{r_2-1},b_{r_2+1},...,b_{r_1-1}\}$, and if $r_1>r_2$, then let $b_{r_2}\rhd b$ for all $b\in\{b_{r_1+1},...,b_{r_2-1},b_{r_2+1},...,b_m\}$. Iterate in this fashion by considering an arbitrary triangle from the remaining partition $\triangle^1\backslash\{\Re^{b_{r_1}},\Re^{b_{r_2}}\}$. At the end of this finite procedure (consisting of exactly m steps), we obtain a transitive but possibly incomplete strict precedence order $\rhd$ on $\mathbf{B}$: Consider an arbitrary strict completion of $\rhd$. It is easy to verify that the adjacency requirement is compatible with the constructed binary relation $\rhd$. Namely, we have $f_{\ell,j}\in\{f_{\ell-1,j},f_{\ell,j+1}\}\subset\mathbf{B}$ whenever $\ell>j$, and $f_{\ell,j}=f_{\ell-1,j}$ if $f_{\ell-1,j}\rhd f_{\ell,j+1}$ and $f_{\ell,j}=f_{\ell,j+1}$ otherwise.

Next, we prove (1) $\Rightarrow$ (3). Suppose that f satisfies the adjacency condition in part (iii) of Theorem 1. Let $b_k=\underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}}\rhd$ where $j<\ell$ and $\mathbf{B}_{\ell j}=\{b_j,...,b_\ell\}$. By adjacency, any entry in the rectangle $\Re^{b_k}_{\ell,j}\ni f_{\ell,j}$ must be a bundle from $\mathbf{B}_{\ell j}$. Because b_k has the highest precedence rank over $\mathbf{B}_{\ell j}$, adjacency implies $f_{k,j'}=b_k$ for all $j\le j'<k$ and $f_{\ell',j}=b_k$ for all $k<\ell'\le\ell$. Thus, $f_{\ell,j}=b_k$.

Finally, we show that (3) $\Rightarrow$ (1). Suppose $f_{\ell,j}=\underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}}\rhd$ whenever $j<\ell$. Clearly $\mathbf{B}_{\ell j}=\mathbf{B}_{(\ell-1)j}\cup\mathbf{B}_{\ell(j+1)}$. Then $\underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}}\rhd=\underset{\{f_{\ell-1,j},f_{\ell,j+1}\}}{\boldsymbol{max}}\rhd$, where we have $f_{\ell-1,j}=\underset{\mathbf{B}_{(\ell-1)j}}{\boldsymbol{max}}\rhd$ and $f_{\ell,j+1}=\underset{\mathbf{B}_{\ell(j+1)}}{\boldsymbol{max}}\rhd$ by (3). Namely, we have $f_{\ell,j}\in\{f_{\ell-1,j},f_{\ell,j+1}\}$, and $f_{\ell,j}=f_{\ell-1,j}$ if $f_{\ell-1,j}\rhd f_{\ell,j+1}$ and $f_{\ell,j}=f_{\ell,j+1}$ otherwise. The precedence order $\rhd$ is strict by (3) because the max operator always has a unique value. If it is not transitive, one can easily construct a transitive (and complete) precedence order $\rhd'$ by using the adjacency of f, where $\underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}}\rhd=\underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}}\rhd'=f_{\ell,j}$ for all $\ell,j\in M$ with $j<\ell$. This completes the proof.

Proof of Theorem 3: Clearly, a C rule belongs to the adjacent rule family. To see that the rank variance of a C rule is lower than any other member of the adjacent rule family, we simply consider two cases about the number of possible alternatives. First, when m is odd, $var(b_n) = (m+1)^2$. For any $b_{n-t}, b_{n+t} \in \mathbf{B}$ with $t < n$, we have $var(b_{n-t}) = var(b_{n+t}) = 2(\frac{(m+1)}{2} - t)^2 + 2(\frac{(m+1)}{2} + t)^2 = (m+1)^2 + 4t^2$. Thus, $var(b_n) < var(b)$ for any $b \in \mathbf{B} \setminus \{b_n\}$.

Since any member of the adjacent rule family must pick an element of $\mathbf{B}$ whenever there is a mutually acceptable alternative in issue X (by cases (ii) and (iii) of Theorem 1), minimization of rank variance requires that $b_n \rhd b$ for any $b \in \mathbf{B} \setminus \{b_n\}$. Also observe that $var(b_n) < var(b_{n-1}) < \ldots < var(b_1)$ and $var(b_n) < var(b_{n+1}) < \ldots < var(b_m)$. Thus, minimization of rank variance subsequently requires that $b_{n-1} \rhd \ldots \rhd b_1$ and $b_{n+1} \rhd \ldots \rhd b_m$. By case (i) of Theorem 1, the outcome for issue X is fixed to o_X whenever there is no mutually acceptable alternative in this issue. Therefore, (o_X, y_n) is the rank minimizing bundle. Note that when m is odd, rank variance of the unique C rule is strictly less than any other member of the adjacent rule family.

On the other hand, when m is even, $var(b_{\overline{n}}) = var(b_{\underline{n}}) = \frac{1}{2}(m^2 + (m+2)^2)$. For any $b_{\underline{n}-t}, b_{\overline{n}+t} \in \mathbf{B}$ with $t < n$, we have $var(b_{\underline{n}-t}) = var(b_{\overline{n}+t}) = 2(\frac{m}{2} - t)^2 + 2(\frac{(m+2)}{2} + t)^2 = \frac{1}{2}(m^2 + (m+2)^2) + 4t^2$. Hence, $var(b_{\overline{n}}) = var(b_{\underline{n}}) < var(b)$ for any $b \in \mathbf{B} \setminus \{b_{\overline{n}}, b_{\underline{n}}\}$. Note that we also have $var(b_{\underline{n}}) = var(b_{\overline{n}}) < var(b_{\underline{n}-1}) < \ldots < var(b_1)$ and $var(b_{\underline{n}}) = var(b_{\overline{n}}) < (b_{\overline{n}+1}) < \ldots < var(b_m)$. Then, minimization of rank variance subsequently requires that either $b_{\overline{n}} \rhd b_{\underline{n}}$ or $b_{\underline{n}} \rhd b_{\overline{n}}$ together with $b_{\underline{n}-1} \rhd \ldots \rhd b_1$ and $b_{\overline{n}+1} \rhd \ldots \rhd b_m$. By case (i) of Theorem 1, the outcome for issue X is o_X and both $(o_X, y_{\overline{n}})$ and $(o_X, y_{\underline{n}})$ are rank minimizing bundles. Consequently, any one of the four types of C rules are rank minimizing. Note that when m is even, rank variance of a C rule is weakly less than any other member of the adjacent rule family.

Proof of Theorem 4:

Proof of 'if' part:

Suppose that Λ is consistent, satisfies weak logrolling and the mediation rule f satisfies following:

(i) If $\ell < j$, then $f_{\ell,j} = (o_X, y)$ for some $y \in Y$.

(ii) If $\ell = j$, then $f_{\ell,j} = (x_\ell, y_{m+1-\ell})$.

(iii) (Adjacency) If $\ell > j$, then $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ and

$$f_{\ell,j} = \begin{cases} f_{\ell-1,j}, & \text{if } f_{\ell-1,j} \rhd f_{\ell,j+1} \\ f_{\ell,j+1}, & \text{oth.} \end{cases}$$

where $\rhd$ is a complete, transitive and antisymmetric precedence order on $\mathbf{B}$ that is admissible with respect to Λ. Then, we want to prove that f is efficient, individually rational and strategy-proof.

It is relatively easy to verify that an adjacent rule f is individually rational: It never suggests an alternative for an issue that is worse than the outside option of that issue, and thus, it is individually rational by the regularity of preferences. To show efficiency, first consider the type profile where both negotiators deem all alternatives acceptable in issue X. At that profile, an adjacent rule proposes a bundle from the set of logrolling bundles $\mathbf{B}$. Let us call this bundle b. If instead the negotiators receive another bundle from $\mathbf{B}$ at that profile, one of the negotiators will certainly get worse off. Suppose for a contradiction that there is another logrolling bundle, say, a in which a is unambiguously better than b for both negotiators, namely $a \ R_i \ b$ for $i = 1, 2$ and for all admissible preferences R_i. On the other hand, transitivity of $\rhd$ implies $b \rhd a$ because the adjacent rule f suggests the bundle b when both a and b are mutually acceptable. Furthermore, because Λ satisfies weak logrolling and $\rhd$ is admissible with respect to Λ, $b \rhd a$ implies either $b \ R_1 \ a$ or $b \ R_2 \ a$ for all admissible preferences (as $\rhd$ concatenates negotiators preferences), contradicting with the presumption that bundle a is unambiguously better than b for both negotiators.

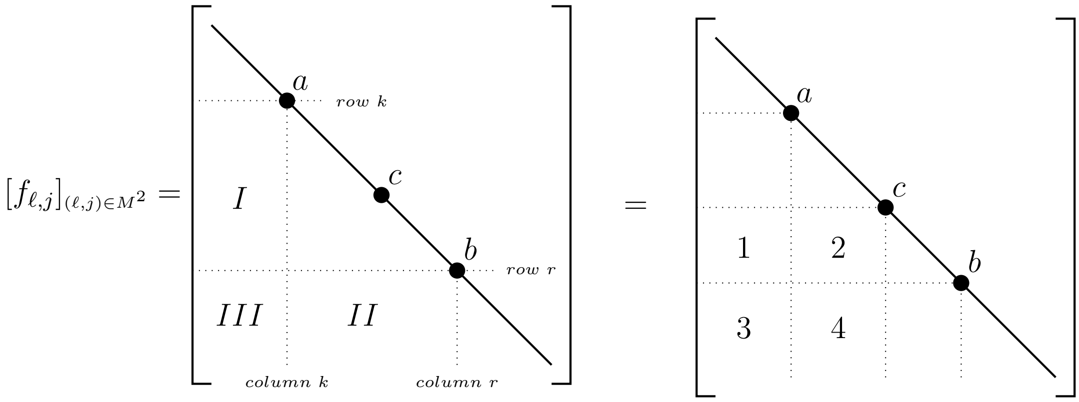
At that type profile, if the negotiators receive a bundle with the outside option in issue X, rather than b, then both negotiators would get worse off because of the deal-breakers assumption. Finally, if the negotiators would receive any other bundle, say c, which is neither a logrolling bundle nor a bundle with the outside option in issue X, then there exists at least one negotiator, i, and an admissible preference ordering, R_i, such that $b \ P_i \ c$, namely the bundle c makes negotiator i worse off at some admissible preference ordering. This is true because neither regularity nor weak logrolling assumption puts a restriction on how negotiators compare bundle b with c.

Thus, no other bundle makes one negotiator better off without hurting the other when both of the negotiators deem all alternatives acceptable. We can directly apply the same logic to all type profiles that the negotiators deem less alternatives acceptable. Finally, for those type profiles where there is no mutually acceptable alternative in issue X, in which case the rule suggests (o_X, y) for some $y \in Y \setminus \{o_Y\}$, any other bundle will include an alternative that is unacceptable in issue X by at least one of the negotiators because their preferences over each individual issue are diametrically opposed. Thus, by regularity at least one negotiator would get worse off if f proposes something other than (o_X, y). Hence, the adjacent rule f is efficient.

We next prove that adjacent rules are strategy-proof, but first establish some facts about the structure of these rules. Let $a = f_{\ell,j}$ and $b = f_{r,s}$ be two bundles, namely bundle a appears on row ℓ and column j whereas bundle b appears on row r and column s. We say bundle a appears above (below) bundle b whenever $\ell < r$ ($\ell > r$). Likewise, bundle a appears the right (left) of bundle b whenever $j > s$ ($j < s$).

Given a mediation rule f and a bundle a that appears on the main diagonal, i.e.,

$a = f_{k,k}$ for some $k \in M$, define $V(a)$ to be the **value region of bundle** a, which is the sub-matrix of $[f_{\ell,j}]_{(\ell,j)\in M^2}$ excluding all the rows lower than row k and all the columns higher than column k. Namely, $V(a) = [f_{\ell,j}]_{(\ell,j)\in(M^k,M_k)}$ where $M^k = \{k, ..., m\}$ and $M_k = \{1, ..., k\}$. Furthermore, if bundle $b = f_{r,r}$ appears on the main diagonal with $r \in M$ and $r > k$, then $V(a) \cap V(b) = [f_{\ell,j}]_{(\ell,j)\in(M^r,M_k)}$ where $M^r = \{r, ..., m\}$. In the following figure, the value region of bundle a is region I and III, value region of bundle b, $V(b)$, is region II and III, and $V(a) \cap V(b)$ is region III.



Lemma 1. *If the mediation rule f is an adjacent rule that is described in Theorem 1, then for any two bundles $a, b \in \mathbf{B}$*

(i) a never appears outside of its value region $V(a)$,

(ii) a and b both never appear in $V(a) \cap V(b)$, and

(iii) if both a and b appear on the same column (or row), where a is above b (or a is on the left of b), then on the main diagonal, bundle a appears above bundle b.

Proof. The first claim directly follows from the last two conditions of Theorem 1. The existence of complete, transitive and strict order $\rhd$ on $\mathbf{B}$ implies the second claim but deserves a proof. Suppose first that a and b appear on the same column in region III, say column s, and a is located above bundle b on this column, namely a is on row r_a and b is on row r_b where $r \leq r_a < r_b \leq m$. Starting from column and row r, i.e., from bundle b, as we move from column r to column s along the row r, adjacency and transitivity of $\rhd$ imply that the bundles on the row r are either ranked higher than b (with respect to $\rhd$) or equal to b, which includes the bundle $f_{r,s}$. Now starting from column s and row r, i.e., the bundle $f_{r,s}$, and move towards row r_a along column s. Adjacency and transitivity of $\rhd$ imply that the bundle on the row r_a and column s, i.e., the bundle a is ranked higher than b with respect to $\rhd$. Namely, $a \rhd b$ must hold.

Continue iterating from where we left. Starting from column s and row r_a, i.e., the bundle a, as we move from row r_a to r_b along the column s, adjacency and transitivity of $\rhd$ imply that all the bundles are either ranked above a or equal to a, including the bundle at row r_b, i.e., b. Thus, we must have $b \rhd a$, contradicting with the fact that $\rhd$ is strict. If bundle b is above bundle a on column s, then we start the iteration from $f_{k,k} = a$. Therefore, a and b cannot appear on the same column in region III. Symmetric arguments suffice to show that they cannot appear on the same row in region III either.

Therefore, suppose that a and b appear on different rows and columns. With similar arguments above, if we start iteration from $f_{r,r} = b$ and go left on the same row and then go down to bundle a in region III, we conclude that $a \rhd b$ by adjacency and transitivity of $\rhd$. However, when we start iteration from $f_{k,k} = a$ and go down on the same column and then go left to bundle b in region III, we conclude that $b \rhd a$, which yields the desired contradiction. Hence, either bundle a or b, whichever is ranked first with respect to $\rhd$, may appear in region III, but not both.

The proof of condition (iii) uses (ii). Suppose for a contradiction that a and b appear on the same column s, where b is above a (i.e., $r_b < r_a$) and a appears above b on the main diagonal. If we refer back to the previous figure, a and b can appear on the same column with $r_b < r_a$ only in region III, which contradicts with what we just proved above. We can make symmetric arguments for rows as well. □

We now ready to show that an adjacent rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$ is strategy-proof. Consider, without loss of generality, deviations of negotiator 1 only. If $\ell < j$, then $A(\theta_1^{x_\ell}, \theta_2^{x_j}) = \emptyset$. Negotiator 1 may receive a different bundle by deviating to a type that is represented by a higher (numbered) row, say $\theta_1^{x_k}$ where $k > \ell$. $A(\theta_2^{x_j})$ is fixed because negotiator 2's type is fixed. Because the negotiators' preferences over issue X are diametrically opposed and f is individually rational, the alternative in issue X at type profile $(\theta_1^{x_k}, \theta_2^{x_j})$ will be unacceptable for negotiator 1's true type, $\theta_1^{x_\ell}$. Thus, by deal-breakers property, negotiator 1 has no profitable deviation from a type profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ with $\ell < j$.

On the other hand, if $\ell = j$, then negotiator 1 can deviate to (1) a lower row and receive (o_X, y), which is worse than $f_{\ell,i} = (x_\ell, y_{m-\ell+1})$ by deal-breakers, or (ii) a higher row and receive a bundle that suggests an unacceptable alternative in issue X. Thus, deal-breakers property imply that negotiator 1 has no profitable deviation in that case either.

Finally, suppose that $\ell > j$. Let $c \in \mathbf{B}$ denote the bundle negotiators get under truthful reporting. If negotiator 1 deviates to a row where f takes the value (o_X, y), then he clearly gets worse off by deal-breakers property. If he deviates to a lower numbered row and receives bundle, say, a, then a appears above bundle c on the first diagonal, by the third condition of Lemma 1. Therefore, we must have $c \;\rhd\; a$ because f suggests c

at some type profile where both a and c are acceptable and f is an adjacent rule. The fact that a appears above bundle c on the first diagonal and $\rhd$ concatenates negotiators preferences imply that $c\ R_1\ a$ for all admissible R_1. Thus, there is no profitable deviation for negotiator 1 by declaring a lower numbered row and getting a instead of c. However, if he declares a higher numbered row and gets a different bundle, say, b, then c appears on the first diagonal above bundle b, again by the third condition of Lemma 1. As it is clearly visible in the last figure, Lemma 1 implies that negotiator 1's true preferences must give him the bundle c in region 1 or 2 and the deviation bundle b must be in region 3 or 4 because they cannot coexists in region 3 or 4. However, bundle b includes alternative x_r from issue X, which is an unacceptable alternative for all types that lie above row r, including negotiator 1's true type. Thus, by deal-breakers property, negotiator 1 has no profitable deviation in that case either. Hence, f is strategy-proof.

***Proof of 'only if'*:** Now suppose that Λ is consistent and the mediation rule f is efficient, individually rational and strategy-proof. We want to prove that Λ satisfies weak logrolling and f satisfies following:

(i) If $\ell < j$, then $f_{\ell,j} = (o_X, y)$ for some $y \in Y$.

(ii) If $\ell = j$, then $f_{\ell,j} = (x_\ell, y_{m+1-\ell})$.

(iii) (Adjacency) If $\ell > j$, then $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ and

$$f_{\ell,j} = \begin{cases} f_{\ell-1,j}, & \text{if } f_{\ell-1,j} \rhd f_{\ell,j+1} \\ f_{\ell,j+1}, & \text{oth.} \end{cases}$$

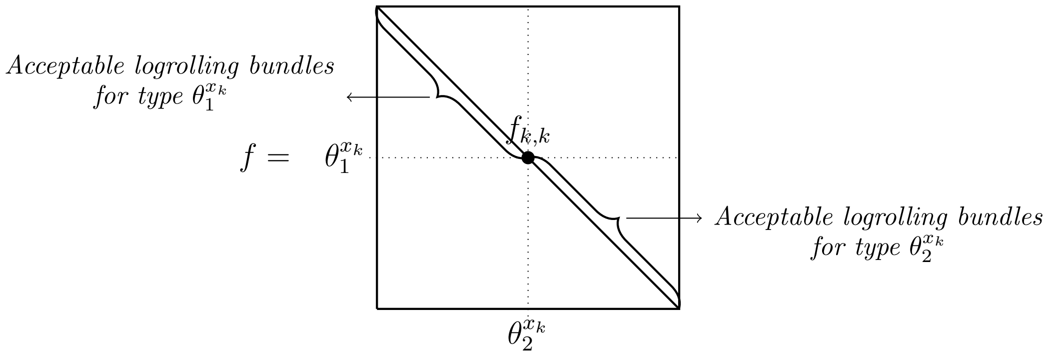
where $\rhd$ is a complete, transitive and antisymmetric precedence order on $\mathbf{B}$ that is admissible with respect to Λ.

***Proof of Part i*:** By individual rationality and regularity of preferences, the alternative for issue X must be o_X whenever $\ell < j$. Then by regularity, $f_{\ell,j} = (o_X, y)$ for some $y \in Y \setminus \{o_Y\}$. By strategy-proofness and monotonicity, we must have $f_{\ell',j} = (o_X, y)$ for all $\ell' < j$. Similarly, $f_{\ell,j'} = (o_X, y)$ for all $\ell < j'$. Fixing j (and ℓ) and applying the same argument for all remaining rows and columns yield $f_{\ell,j} = (o_X, y)$ whenever $\ell < j$.

***Proof of Part ii*:** Consider the main diagonal where $\ell = j = k$. Now, we want to show that $f_{k,k} = (x_k, \hat{y}_k)$ for every $k = 1, ..., m$ and $\hat{y}_k = y_{m+1-k}$. Row and column k correspond to preference profile $(\theta_1^{x_k}, \theta_2^{x_k})$ where the only mutually acceptable alternative in issue X is x_k. Therefore, for any $1 \leq k \leq m$ efficiency and individual rationality of f and regularity of preferences imply $f^X_{k,k} = x_k$ and $f^X_{k+1,k} \in \{x_k, x_{k+1}\}$ whenever $k \neq m$. We claim that $f^Y_{k+1,k+1}\ \theta_1^Y\ f^Y_{k,k}$ for each $k = 1, ..., m-1$. If this statement is correct, then we have the desired result because the number of alternatives in issue X and Y are the same.

Consider any $1 \leq k \leq m-1$. If $f^X_{k+1,k} = x_{k+1}$, then strategy-proofness and monotonicity of preferences of negotiator 2 imply that $f^Y_{k+1,k} = f^Y_{k+1,k+1}$. Similarly, strategy-proofness and monotonicity of preferences of negotiator 1 requires that $f^Y_{k+1,k+1}\ \theta^Y_1\ f^Y_{k,k}$. On the other hand, if $f^X_{k+1,k} = x_k$, then strategy-proofness and monotonicity of preferences of negotiator 1 imply that $f^Y_{k+1,k} = f^Y_{k,k}$. But then strategy-proofness and monotonicity of preferences of negotiator 2 requires that $f^Y_{k,k}\ \theta^Y_2\ f^Y_{k+1,k+1}$, which implies $f^Y_{k+1,k+1}\ \theta^Y_1\ f^Y_{k,k}$ as the negotiators' preferences over the alternatives in issue X are diametrically opposed.

REMARK: An important implication of part (ii) is that for type $\theta^{x_k}_2$ of negotiator 2 all the logrolling bundles in $\mathbf{B}$ that appear below the bundle $f_{k,k}$ on the first diagonal are acceptable, i.e., strictly better than the bundle of outside options. Likewise, for type $\theta^{x_k}_1$ of negotiator 1 all the logrolling bundles in $\mathbf{B}$ that appear above the bundle $f_{k,k}$ on the first diagonal are acceptable.



***Proof of Part iii*:** We refer to bundles $\{f_{k,1}, f_{k+1,2}, ..., f_{m,m-k+1}\}$ where $k = 1, ..., m$ as those on the $k-$th diagonal. Note that each diagonal has one less bundle than its immediate predecessor and the $m-$th diagonal consists of a single bundle, namely $f_{m,1}$.

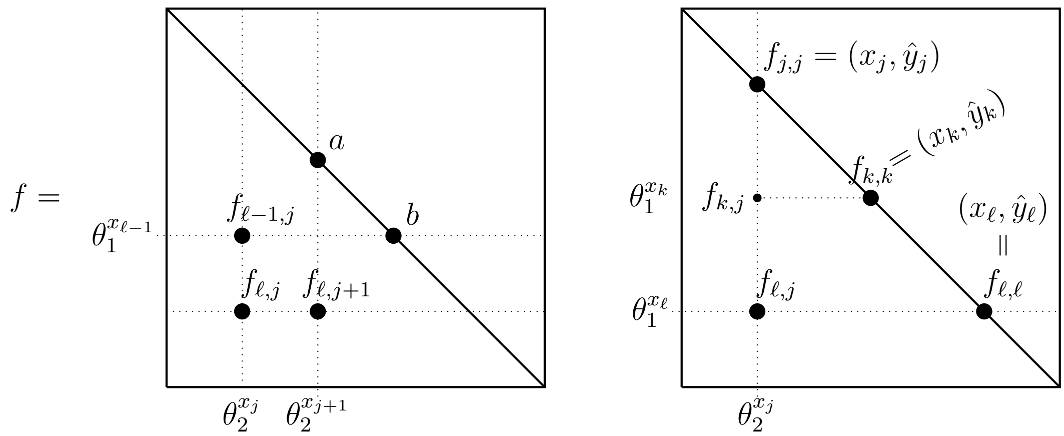
Lemma 2. *Suppose that adjacency holds for all bundles on all diagonals* $t = 2, ..., k$ *where* $k \leq m$*. That is, for all* $t \in \{2, ..., k\}$ *and* $m \geq \ell > j$ *with* $\ell = j + t - 1$*,* $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$*. Consider two bundles* $a, b \in \mathbf{B}$ *that appear on some diagonal* $t \in \{2, ..., k\}$*. If bundle* a *lies on a higher row than* b *on the first diagonal, then* a *also lies on a higher row than* b *on all diagonals up to (and including) diagonal* t*.*

Proof. Since both a and b appear on diagonal t, by adjacency, they both must also appear on every diagonal from the second through $(t-1)-$st diagonal. Suppose that a lies above b on the first diagonal. From the first diagonal to the second, adjacency implies that a bundle can either move by one cell horizontally to the left or drop by one cell down. If a moves horizontally, clearly it will remain above b on the second diagonal. If a drops by one cell, it remains above b or on the same row with b (which happens when a and b are diagonally adjacent on the first diagonal). In the former case, b is clearly below a on the second diagonal. In the latter case, for b to also appear on the second diagonal it must also have dropped one cell below, in which case it is again below a on the second diagonal. Iterating this argument for rows 3 through t yields the desired result. $\square$

STEP 1 (Adjacency): We first show the following: Take a bundle on some diagonal except the first one. This bundle is equal to the bundle immediately above it or immediately to its right. Lemma 3 states this more formally.

Lemma 3. *$f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ for all $j < \ell \leq m$.*

Proof. In the proof of part (ii) we showed that the set of bundles on the first diagonal is equal to the set of logrolling bundles, $\mathbf{B}$. Consider any $j < \ell \leq m$ and suppose for a contradiction that $f_{\ell,j} \notin \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$. Note that strategy-proofness implies $f_{\ell,j}\ R_1\ f_{\ell-1,j}$ for all admissible R_1 and all types that find bundles $f_{\ell,j}$ and $f_{\ell-1,j}$ acceptable, and $f_{\ell,j}\ R_2\ f_{\ell,j+1}$ for all admissible R_2 and all types that find bundles $f_{\ell,j}$ and $f_{\ell,j+1}$ acceptable.



The bundles $f_{\ell,j+1}$ and $f_{\ell-1,j}$ appear on the first diagonal because they both are in $\mathbf{B}$. Furthermore, $f_{\ell,j+1}$ appears on or below bundle a while $f_{\ell-1,j}$ appears on or above bundle b because f is individually rational (see the last figure). If $f_{\ell,j} \in \mathbf{B}$, then it should also appear on the main diagonal. If it is located below a, then it means type $\theta_2^{x_{j+1}}$ finds both $f_{\ell,j}$ and $f_{\ell,j+1}$ acceptable, which means she has incentive to deviate to $\theta_2^{x_j}$ to get $f_{\ell,j}$, contradicting with strategy-proofness of f. On the other hand, if it is located above b, then type $\theta_1^{x_{\ell-1}}$ finds both $f_{\ell,j}$ and $f_{\ell-1,j}$ acceptable, meaning that she has incentive to deviate to $\theta_1^{x_\ell}$ to get $f_{\ell,j}$, contradicting again with strategy-proofness of f. Thus, $f_{\ell,j}$ has to be equal to one of these two bundles if $f_{\ell,j}$ is a logrolling bundle.

Suppose now that $f_{\ell,j} \notin \mathbf{B}$. Note that row ℓ and column j correspond to the profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ and by efficiency and individual rationality $f^X_{\ell,j} \in \{x_j, x_{j+1}, ..., x_\ell\}$. By monotonicity and strategy-proofness $f^X_{\ell,j}$ cannot be x_j because $f_{\ell,j} \notin \mathbf{B}$ and so $f^Y_{\ell,j} \neq \hat{y}_j$. Similarly, $f^X_{\ell,j} \neq x_\ell$. Now suppose that $f^X_{\ell,j} = x_k$ where $j < k < \ell$. Again by monotonicity and strategy-proofness we must have $f_{k,j} = f_{\ell,j}$: this is true because any bundle acceptable by type $\theta_1^{x_k}$ are also acceptable by type $\theta_1^{x_\ell}$ and $f_{\ell,j}$ is acceptable by $\theta_1^{x_k}$, and thus if $f_{k,j} \neq f_{\ell,j}$ one of these types would have incentive to deviate. But then again by monotonicity and strategy-proofness (regarding negotiator 2) we have $f_{k,j} = f_{\ell,j} = f_{k,k}$, which contradicts with the presumption that $f_{\ell,j} \notin \mathbf{B}$. □

STEP 2 (Construction of a precedence order $\rhd$): By step 1, we know that $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ for all $\ell > j$. To construct $\rhd$, perform a pairwise comparison for all the entries $f_{\ell,j}, f_{\ell-1,j}, f_{\ell,j+1}$. More formally, $f_{\ell-1,j} \ \rhd \ f_{\ell,j+1}$ whenever $f_{\ell,j} = f_{\ell-1,j}$ and $f_{\ell,j+1} \ \rhd \ f_{\ell-1,j}$ whenever $f_{\ell,j} = f_{\ell,j+1}$. We obtain a partial order $\rhd$ on $\mathbf{B}$, which may not be complete at this point. Next, we will show that $\rhd$ is antisymmetric and transitive.

Lemma 4. *Order $\rhd$ is antisymmetric. That is, for any $a, b \in \mathbf{B}$, $a \rhd b$ implies $\neg b \rhd a$.*

Proof. Suppose for a contradiction that there is $a, b \in \mathbf{B}$ such that both $a \rhd b$ and $b \rhd a$ hold. Let $t \geq 1$ be the smallest diagonal on which a and b are diagonally adjacent and a is "chosen" according to $\rhd$. That is, let $f_{\ell-1,j} = a$, $f_{\ell,j+1} = b$, and so $f_{\ell,j} = a$. Because f is efficient, individually rational and taking values a and b when negotiator 1 announces his type as $\theta_1^{x_\ell}$, both bundles must be acceptable for all types $\theta_1^{x_k}$ where $k \geq \ell$. Moreover, because $b \rhd a$ by assumption, there must exist another diagonal $t' > t$ in which a and b are diagonally adjacent and b is chosen. By Lemma 2, bundles a and b cannot be adjacent to one another more than once on the same diagonal, and thus $t' > t$. Therefore, let $f_{s-1,r} = a$, $f_{s,r+1} = b = f_{s,r}$. By Lemma 2 and Lemma 3, we have $s > \ell$. Strategy-proofness implies that $b \ R_1 \ a$ for all admissible $R_1 \in \Lambda(\theta_1^{x_s})$, and so we must have $b \ R_1 \ a$ for all admissible $R_1 \in \Lambda(\theta_1^{x_k})$ where $k \geq \ell$, including type $\theta_1^{x_{s-1}}$. But the last observation contradicts with strategy-proofness of f as type $\theta_1^{x_{s-1}}$ would profitably deviate to $\theta_1^{x_s}$ and get bundle b rather than a. □

Let two bundles a and b be diagonally adjacent. If a lies on a higher row than b, then we say that a is *diagonally adjacent to b from below*. Equivalently, we say that b is *diagonally adjacent to a from above*.

Lemma 5. (*i*) *Let bundle $a = f_{\ell,j} \in \mathbf{B}$ be diagonally adjacent to some bundle $b \in \mathbf{B}$ from below and $a \rhd b$. Then, bundle b never appears on or below row ℓ, i.e., there is no $k \geq \ell$ and r such that $f_{k,r} = b$. Additionally, bundle a never appears (strictly)* *above row ℓ and (strictly) to the left of column j, i.e., there is no $\ell' < \ell$ and $j' < j$ such that $f_{\ell',j'} = a$.*

(*ii*) *Let bundle $c = f_{\ell,j} \in \mathbf{B}$ be diagonally adjacent to some bundle $d \in \mathbf{B}$ from above and $c \rhd d$. Then, bundle d never appears on column j or any lower column, i.e., there is no $k \leq j$ and r such that $f_{r,k} = d$. Additionally, bundle c never appears (strictly) below row ℓ and (strictly) to the right of column j, i.e., there is no $\ell' > \ell$ and $j' > j$ such that $f_{\ell',j'} = c$.*

Proof. We prove the first part, i.e., (i), as symmetric arguments will suffice to prove part (ii). First part of (i): The bundle b must be above a on the first diagonal because b is above a at some diagonal. Moreover, negotiator 2 may receive bundles a and b (depending on negotiator 1's type) when he declares his type as $\theta_2^{x_{j-1}}$, and so by efficiency and individual rationality of f, both these bundles must be acceptable for type $\theta_2^{x_{j-1}}$ of negotiator 2. Suppose for a contradiction that b occurs below row ℓ. By the adjacency property, this b should be coming all the way from the main diagonal, and so b must also appear on row ℓ. Let $f_{\ell,k} = b$ for some $k \neq j, j-1$. Strategy-proofness implies $b\ R_2\ a$ for all admissible R_2 and all types of negotiator 2 that deem both bundle a and b acceptable. But then type $\theta_2^{x_{j-1}}$ would profitably deviate to type $\theta_2^{x_k}$ in order to get the bundle b rather than a, contradicting strategy-proofness of f.

Second part of (i): Because $f_{\ell-1,j-1} = b$, the bundle b must appear on the first diagonal on column j or higher. Because a is below b on main diagonal as well, it also can appear on the main diagonal on column $j+1$ or higher. Therefore, if bundle a appears in the region, for a contradiction, then by the adjacency property bundle a must appear on column $j+1$ as well. Let $f_{k,j+1} = a$ for some $k \leq \ell - 1$. But if a is acceptable for type $\theta_1^{x_k}$ of negotiator 1, it must also be acceptable for type $\theta_1^{x_{\ell-1}}$ of negotiator 1, when he gets the bundle b. Therefore, because $f_{\ell,j-1} = a$ and $f_{\ell-1,j-1} = b$, strategy-proofness implies $b\ R_1\ a$ for all admissible preferences and all types of negotiator 1 that deem both bundles acceptable. But then type $\theta_1^{x_\ell}$ deems both bundles acceptable and prefers to deviate to type $\theta_1^{x_{\ell-1}}$ to get b rather than a, contradicting strategy-proofness of f. $\square$

Lemma 6. *Order $\rhd$ is transitive. That is, for any triple $a, b, c \in \mathbf{B}$ such that $a \rhd b$ and $b \rhd c$, we have $\neg c \rhd a$.*

Proof. Suppose, for a contradiction, that $a \rhd b$ and $b \rhd c$, but $c \rhd a$. Without loss of generality, suppose b is diagonally adjacent to a from above. Let $t \geq 1$ be the smallest diagonal on which a and b are adjacent where $f_{\ell,j} = a$, $f_{\ell-1,j-1} = b$ and $f_{\ell,j-1} = a$ because $a \rhd b$. By Lemma 5 part (i), b never appears on row ℓ or below. Let t' be the smallest diagonal on which b and c are adjacent. We consider two cases:

Case 1: $t' \geq t$: This case has two subcases:

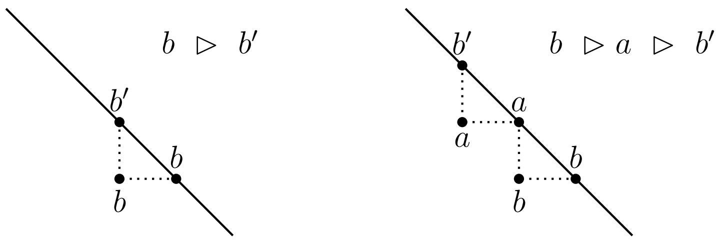
Case 1A: Suppose first that c is adjacent to b from below on diagonal t': Consider diagonal t. Clearly, c should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t' \geq t$. Then by Lemma 3, since c is adjacent to b from below on diagonal t', it must appear below b on row $\ell + 1$ or below on diagonal t. Then by Lemma 2 and adjacency, c can appear only on $\ell + 1$ or below on diagonal $t' \geq t$ as well. However, By Lemma 5 part (i), $f_{\ell,j} = a \rhd b$ implies that b can never appear on row ℓ or below. This means b and c cannot be adjacent on diagonal $t' \geq t$; a contradiction.

Case 1B: Suppose now that c is adjacent to b from above on diagonal t'. Let $f_{p,q} = b$ and $f_{p-1,q-1} = c$. Because b never appears on row ℓ or below, $p \leq \ell - 1$. By Lemma 4, $b \rhd c$ implies $f_{p,q-1} = b$. By Lemma 5 part (i), $b \rhd c$ implies that c never appears on row p or below. Because b is diagonally adjacent to a from above and c is adjacent to b from above, by Lemma 3, $c \rhd a$ implies that c must be adjacent to a from above on some diagonal t''. By Lemma 3, there is no b on diagonal t'' for otherwise it would be either below a or above c. Then $t'' > t'$. Thus, let $f_{r,s} = a$ and $f_{r-1,s-1} = c$ on diagonal t'', and so $f_{r,s-1} = c$ by $c \rhd a$. Because there is no c on or below row $p \leq \ell - 1$, a and c must then be adjacent above row p on diagonal $t'' > t'$. That is, $r < p$. Then $t'' > t'$ implies that $s \leq q - 2$. Because there is no c on row $p \leq \ell - 1$ or below and $t' \geq t$, $f_{r,s} = a$ lies on row above row p, i.e., $r < p$ and on column $j - 2$ or to the left, i.e., $s \leq j - 2$. However, by Lemma 5 part (i), $f_{\ell,j} = a \rhd b$ implies that bundle a should never appear in the box (strictly) above row ℓ and (strictly) to the left of column j; a contradiction.

Case 2: $t' < t$: This case also has two subcases.

Case 2A: Suppose c is adjacent to b from above on diagonal t'. Consider diagonal t'. Clearly, a should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t > t'$. Since a lies below b on diagonal t, it must again be below b on diagonal t'. Let k be the row on which b lies on diagonal t'. Clearly, a lies below row k on diagonal t' or any other diagonal $t'' > t$. Since c is adjacent to b from above on diagonal t' and $b \rhd c$, Lemma 5 part (i) implies that c never appears on row k or below. Thus, a and c cannot be diagonally adjacent on any diagonal $t'' > t'$. But they cannot be diagonally adjacent on any diagonal $t''' < t'$ either because that would mean that there is no b on diagonal t''' for otherwise b would be above c or below a, contradicting Lemma 3; a contradiction.

Case 2B: Suppose c is adjacent to b from below on diagonal t'. Consider diagonal t'. Clearly, a should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t > t'$. Because a lies below b on diagonal t, it must lie below both b and c on diagonal t'. Suppose a and c are diagonally adjacent on some diagonal t''. Let $f_{p,q} = c$ on diagonal t''. Clearly, c must lie above a on diagonal t''. Because b is diagonally adjacent to a from above on diagonal t, there is no c on diagonal t (or on any higher numbered diagonal) for otherwise c would be above b or below a on diagonal t, contradicting Lemma 2. Thus, $t'' < t$. Since $a = f_{p+1,q+1}$ and $c = f_{p,q}$ are diagonally adjacent on t'' and $c \rhd a$, Lemma 5 part (ii) implies that a never appears on column q or any lower numbered column. Since $f_{\ell,j} = a$, we need $q < j-1$. Since $t'' < t$ and $q < j-1$, bundle $a = f_{p+1,q+1}$ must lie above row ℓ. But, recall that Lemma 5 part (i) and $f_{\ell,j} = a \rhd b$ implies that a should never appear in the box (strictly) above row ℓ and (strictly) to the left of column j; a contradiction. $\square$



Lemma 7. *Order $\rhd$ is admissible with respect to Λ.*

Proof. We need to prove that the order $\rhd$, which we created by using the efficient, individually rational and strategy-proof f, is transitive, antisymmetric and connected binary relation that concatenates negotiators' preferences. We already proved the first two properties. Connectedness of $\rhd$ is simple. Let all logrolling bundles on diagonal k of f constitute the set $B^k_{\rhd}$. Then by construction of $\rhd$, it is complete with respect to adjacency on all $B^k_{\rhd}$ where $k = 1, ..., m$. Thus, $\rhd$ is connected.

To show that $\rhd$ concatenates negotiators' preferences take any $b, b' \in \mathbf{B}$ where b' is located above b on the first diagonal and $b \;\rhd\; b'$ (symmetric arguments work when b is located above b' on the first diagonal). If these two bundles are ever adjacent on some diagonal, then $b \;\rhd\; b'$ means b is located below b' on some column, and so by strategy-proofness we must have $b\ R_1\ b'$ for all admissible R_1's as required. If however, these two bundles are never adjacent but $b \;\rhd\; b'$ is the result of the transitivity of $\rhd$, then adjacency of f, i.e., part (iii), implies that there must exist at least one $a \in \mathbf{B}$ such that a appears on the main diagonal above b and below b', a is adjacent to b on some diagonal and $b \;\rhd\; a$, and a is adjacent to b' on some (other) diagonal and $a \;\rhd\; b'$.

Adjacency of a and b and $b \;\rhd\; a$ imply b is located below a on some column, and so by strategy-proofness we must have $b\ R_1\ a$ for all admissible R_1's. Similarly, adjacency of a and b' and $a \;\rhd\; b'$ imply that a is located below b' on some column, and so by strategy-proofness we must have $a\ R_1\ b'$ for all admissible R_1's. Thus, by transitivity of admissible preferences, we have $b\ R_1\ b'$ for all admissible R_1's, as required. If there were multiple bundles like a in between b and b', then we repeat these transitivity arguments multiple times and reach the same conclusion. $\square$

Finally, we stipulate that any incomplete portions of partial order $\rhd$ are chosen in any arbitrary manner without violating transitivity. This and Lemmas 4-7 give us a complete, transitive and antisymmetric precedence order $\rhd$ that is admissible with respect to Λ.

The Revelation Principle

We prove the revelation principle for the symmetric treatment of the outside options. The same logic applies directly to the case with heterogeneous treatment of the outside options. A mediation mechanism $\Gamma = \big(S_1, S_2, g(.)\big)$ with veto rights is a collection of strategy sets (S_1, S_2) and an outcome function $g : S_1 \times S_2 \to X \times Y$. The mechanism Γ combined with possible types (Θ_1, Θ_2) and preferences over bundles (R_1, R_2) with $R_i \in \Lambda(\theta_i)$ for all i defines a game of incomplete information. A strategy for negotiator i in the game of incomplete information created by a mechanism Γ is a function $s_i : \Theta_i \to S_i$.

Lemma 8 (Revelation Principle in Dominant Strategies). *Suppose that there exists a mechanism $\Gamma = \big(S_1, S_2, g(.)\big)$ that implements the mediation rule f in dominant strategies. Then f is strategy-proof and individually rational.*

Proof. If Γ implements f in dominant strategies, then there exists a profile of strategies $s^*(.) = \big(s_1^*(.), s_2^*(.)\big)$ such that $g\big(s^*(\theta)\big) = f(\theta)$ for all $\theta \in \Theta$, and for all $i \in I$ and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}(\theta_{-i})\big) \; R_i \; g\big(s_i'(\theta_i'), s_{-i}(\theta_{-i})\big) \tag{1}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ and all $s_i'(.), s_{-i}(.)$. Condition 1 must also hold for s^*, meaning that for all i and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big) \; R_i \; g\big(s_i^*(\theta_i'), s_{-i}^*(\theta_{-i})\big) \tag{2}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last inequality implies that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i}) \; R_i \; f(\theta_i', \theta_{-i}) \tag{3}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$.

On the other hand, because the mechanism Γ always allows negotiators to veto proposed bundle before the mediation game ends, there exists a deviation strategy $\hat{s}_i(.)$ for any strategy $s_i(.)$ such that $g(\hat{s}_i(\theta_i), s_{-i}) = \big(o_X, o_Y\big)$ for all $\theta_i \in \Theta_i$ and all $s_{-i} \in S_{-i}$. The idea is that the negotiator i plays in $\hat{s}_i(.)$ exactly the same way in $s_i(.)$ (for all θ_i's) until the ratification stage and vetoes the proposed bundle.

Therefore, if $\hat{s}_i(.)$ is such a deviation strategy for $s_i^*(.)$, then condition 1 must also hold for $\hat{s}_i(.)$, implying that for all i and $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big)\ R_i\ g\big(\hat{s}_i(\theta_i'), s_{-i}^*(\theta_{-i})\big) = \big(o_X, o_Y\big)$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last condition means that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i})\ R_i\ \big(o_X, o_Y\big) \tag{4}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Hence, conditions 3 and 4 imply that f is strategy-proof and individually rational. □

Proof of Proposition 2: Consider the (true) preference profile $(\theta_1, \theta_2) = (\theta_1^{xm}, \theta_1^{ym}, \theta_2^{x_1}, \theta_2^{y_1})$. That is, both negotiators find all alternatives acceptable. Let $(x, y) = f(\theta_1, \theta_2)$. Because negotiators preferences over alternatives are diametrically opposed for each single issue, there is at least one negotiator $i \in$ I and an issue for which negotiator i does not get her top alternative for that issue. Suppose, without loss of generality, that this negotiator is 1 and the issue is X: that is, $x \neq x_1$. Consider the new profile where only negotiator 1's preferences are different, $(\theta_1', \theta_2) = (\theta_1^{x_1}, \theta_1^{y_1}, \theta_2^{x_1}, \theta_2^{y_1})$.

We claim that $f(\theta_1', \theta_2) = (x_1, y_1)$. Suppose for a contradiction that $f(\theta_1', \theta_2) = (x', y') \neq (x_1, y_1)$. I will only show that $x' = x_1$ because similar arguments also prove $y' = y_1$, yielding the desired contradiction. To show $x' = x_1$, suppose for a contradiction that $o_X\ \theta_1^{x_1}\ x'$. Since Λ satisfies DB, $(o_X, o_Y)\ P_1\ (x', y')$ for all $R_1 \in \Lambda(\theta_1')$, and thus $f(\theta_1', \theta_2) = (x', y')$ contradicts with the individual rationality of f. Now suppose for a contradiction that $x' = o_X$. Then, since Λ satisfies Monotonicity, $(x_1, y')\ P_i\ (x', y')$ for $i = 1, 2$ and all $R_1 \in \Lambda(\theta_1^{x_1})$ and all $R_2 \in \Lambda(\theta_2^{x_1})$. Therefore, (x', y') is an inefficient bundle at (θ_1', θ_2), and thus $f(\theta_1', \theta_2) = (x', y')$ contradicts with the efficiency of f. Hence, we must have $x' = x_1$.

To conclude, we already know that $f(\theta_1, \theta_2) = (x, y)$ and $x \neq x_1$, which implies $x_1\ \theta_1^{xm}\ x$. Because y_1 is negotiator 1's best alternative in issue Y, either $y = y_1$ or $y_1\ \theta_1^{y_1}\ y$ is true. In either case, Monotonicity and transitivity of preferences imply $(x_1, y_1)\ P_1\ (x, y)$ for all $R_1 \in \Lambda(\theta_1)$. Finally, we showed in the previous paragraph that by misrepresenting his preferences at profile (θ_1, θ_2), negotiator 1 can achieve the bundle (x_1, y_1), which is strictly better than (x, y) for all $R_1 \in \Lambda(\theta_1)$, contradicting that f is strategy-proof.

Proof of Theorem 5:

Proof of 'if': Same arguments in the proof of Theorem 1 suffice to verify that the mediation rule described in Theorem 5 is individually rational and efficient. Lemma 1 also holds in the continuous case. The proof of part (i) of Lemma 1 is straightforward; given the location of a logrolling bundle a on the main diagonal, $f_{\ell,j}$ can be a only if $a \in \mathbf{B}_{\ell j}$, and so, a can never appear outside of its value region $V(a)$. To prove part (ii), let $f_{\ell,j} = a$ and $f_{s,r} = b$ and suppose for a contradiction that $a, b \in V(a) \cap V(b)$. Therefore, we have $a, b \in \mathbf{B}_{\ell j} \cap \mathbf{B}_{sr}$. The bundle a beats b with respect to $\rhd$ because a wins over $\mathbf{B}_{\ell j}$. Likewise, b beats a with respect to $\rhd$ because b wins over $\mathbf{B}_{sr}$. The last two observations contradict with the assumption that $\rhd$ is strict. To prove part (iii), suppose that $f_{\ell,s} = a$ and $f_{j,s} = b$ where $\ell < j$, whereas a appears below b on the main diagonal. This is possible only when $a, b \in V(a) \cap V(b)$, contradicting with the second part. Similar arguments prove the claim when bundles a and b are on the same row.

Now we prove that f is strategy-proof. It suffices to consider the deviations of one negotiator to prove that f is strategy-proof. Take any $\ell, j \in [0,1]$ such that $f(\theta_1^\ell, \theta_2^j) = f_{\ell,j} = (o_X, y)$ (see figure 6-a). Deviating from θ_1^ℓ does not benefit negotiator 1 if he deviates to θ_1^s where $s < j$ because the outcome of f will not change. However, if negotiator 1 deviates to some $s \geq j$ and get some b, we know that b is one of the logrolling bundles in $\mathbf{B}_{sj}$. However, all of the bundles in $\mathbf{B}_{sj}$ are unacceptable for type θ_1^ℓ of negotiator 1 since $\ell < s$, and so, not preferable to (o_X, y) by the deal-breaker property.

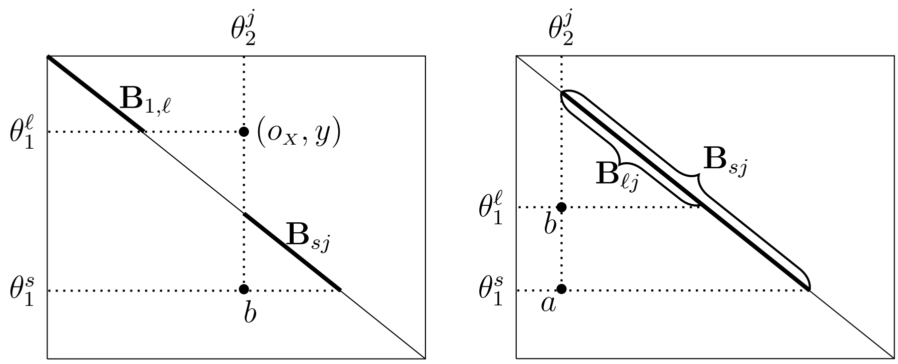


Figure 6-a

Figure 6-b

Now take any $\ell, j \in [0,1]$ such that $\ell \geq j$ and $f(\theta_1^\ell, \theta_2^j) = f_{\ell,j} = b \in \mathbf{B}$. Deviating from θ_1^ℓ does not benefit negotiator 1 if he deviates to θ_1^s where $s < j$ because the outcome of f would be (o_X, y), which is not better than $b \in \mathbf{B}$ by the deal-breaker property. If negotiator 1 deviates to some $\ell > s \geq j$ and get some a, then a must appear above b on the main diagonal (part (ii) of Lemma 1). Logrolling implies that negotiator 1 finds b at least as good as a at all admissible preferences, and thus, deviating to s is not profitable.

Finally, suppose that negotiator 1 deviates to some $s > \ell \geq j$ and get some a (see figure 6-b). Therefore, a beats b with respect to $\rhd$ because both a and b are in $\mathbf{B}_{sj}$ and a is chosen. Thus, a cannot be an element of $\mathbf{B}_{\ell j}$ as b is the maximizer of $\rhd$ over this set. Thus, $a \in \mathbf{B}_{sj} \setminus \mathbf{B}_{\ell j}$, implying that a is not acceptable for type θ_1^ℓ, and so, deviating to θ_1^s is not profitable by the deal-breaker property. Hence, f is strategy-proof.

Proof of 'only if': The same arguments in the proof of Theorem 1 suffices to show that there must exist some $y \in Y \setminus \{o_Y\}$ such that $f_{\ell j} = (o_X, y)$ for all $\ell, j \in [0,1]$ with $\ell < j$. Consider now for $\ell \geq j$.

STEP 1 (Adjacency):

Lemma 9. *If f is a strategy-proof, individually rational and efficient mediation rule, then $f_{\ell,j} \in \mathbf{B}_{\ell j}$ for all $\ell \geq j$.*

Proof. We first show that $f_{kk} = (k, 1-k) \in \mathbf{B}_{kk}$ for all $k \in [0,1]$. Suppose for a contradiction that there is some $\ell \in [0,1]$ such that $f_{\ell,\ell} = (x,y) \neq (\ell, 1-\ell)$, and so $f_{\ell,\ell} \notin \mathbf{B}$. By individual rationality and deal-breaker property, we have $x = \ell$ because ℓ is the only mutually acceptable alternative in X at type profile $(\theta_1^\ell, \theta_2^\ell)$. Next, we show that $f_{\ell,k} = (\ell, y)$ for any $k < \ell$. Suppose not, i.e., there is some $j < \ell$ such that $f_{\ell,j} = (x', y') \neq (\ell, y)$. Individual rationality implies $x' \leq \ell$, and so, there are three exhaustive cases we need to consider:

1. If $x' = \ell$ and $y' \geq y$, then by monotonicity θ_2^ℓ profitably deviates to θ_2^j, contradicting with strategy-proofness.

2. If $x' \leq \ell$ and $y' \leq y$, then by monotonicity θ_2^j profitably deviates to θ_2^ℓ, contradicting with strategy-proofness.

3. If $x' < \ell$ and $y' > y$, then bundles (ℓ, y) and (x', y') are not unambiguously comparable, i.e., there exists an admissible preference ordering of negotiator 1 where the bundle (ℓ, y) is preferred to the bundle (x', y') and another admissible ordering where (x', y') is preferred to (ℓ, y). Therefore, type θ_1^ℓ would profitably deviate to θ_1^j, contradicting again with strategy-proofness.

Thus, we must have $f_{\ell,j} = (\ell, y)$. Given that $f_{\ell,j} = (\ell, y)$, symmetric arguments suffice to prove that $f_{j,j} = (\ell, y)$ as well, which contradicts individual rationality because $\ell > j$ is not acceptable by type θ_1^j of negotiator 1. Thus, we have $y = 1 - \ell$, and so $f_{\ell,\ell} \in \mathbf{B}_{\ell\ell}$.

Now consider the case where $\ell > j$ and suppose for a contradiction that $f_{\ell,j} = (x,y) \notin \mathbf{B}$. By individual rationality we have $x \in [j, \ell]$. Moreover, strategy-proofness implies $x = j$. Suppose not, i.e., $x > j$. If $y \geq 1-j$, then there is an admissible preference ordering of negotiator 1 such that the bundle $f_{j,j} = (j, 1-j)$ is preferred to the bundle $f_{\ell,j} = (x,y)$ by monotonicity, and so type θ_1^ℓ would profitably deviate to type θ_1^j, contradicting with strategy-proofness. On the other hand, if $y < 1 - j$, then bundles $f_{j,j}$ and (x,y) are not unambiguously comparable, namely there exists an admissible preference ordering of negotiator 1 where the bundle $f_{j,j}$ is preferred to the bundle $f_{\ell,j}$ and another admissible ordering where $f_{\ell,j}$ is preferred to $f_{j,j}$. Therefore, type θ_1^ℓ would profitably deviate to θ_1^j, contradicting again with strategy-proofness. Symmetric arguments suffice to prove that strategy-proofness imply $x = \ell$ because otherwise negotiator 2 would profitably deviate. The last two claims lead to the desired contradiction because we must have $x = j$ and $x = \ell$, but $\ell > j$. Finally, given that $f_{\ell,j} \in \mathbf{B}$, individual rationality requires $f_{\ell,j} \in \mathbf{B}_{\ell j}$. $\square$

STEP 2 (Construction of a precedence order): To construct $\triangleright$, we perform the following pairwise comparison: Let $f_{\ell,\ell} = a \in \mathbf{B}$ and $f_{j,j} = b \in \mathbf{B}$ for some $\ell, j \in [0,1]$ with $\ell > j$ and define $a \triangleright b$ whenever $f_{\ell,j} = a$ and $b \triangleright a$ whenever $f_{\ell,j} = b$. The binary relation $\triangleright$ is asymmetric by definition because the logrolling bundles a and b can appear on the main diagonal only once. However, it may not be complete. Lemma 10 below proves that this binary relation is not empty. Namely, there exists some such a and b where either $a \triangleright b$ or $b \triangleright a$.

Lemma 10. *Let the mediation rule f be strategy-proof, individually rational, efficient and $f_{\ell,j} = a \in \mathbf{B}$ where $\ell > j$. Then there exists some $k \geq j$ such that $f_{k,k} = a$ and $f_{\ell,k} = a$.*

Proof. Given that $f_{\ell,j} = a \in \mathbf{B}$ where $\ell > j$, Lemma 9 implies that $a \in \mathbf{B}_{\ell j}$, and so there is some $k \in [j, \ell]$ such that $f_{k,k} = a$. To prove the second part, suppose for a contradiction that $f_{\ell,k} = z$ where $z \neq a$. Again by Lemma 9, we know that $z \in \mathbf{B}_{\ell k}$, and so there is some $k' \in [k, \ell]$ such that $f_{k',k'} = z$. By the way the logrolling bundles are ranked by negotiator 2, $f_{k,k} = a$ is preferred to $f_{k',k'} = z$ because $k < k'$. Therefore, given that the type of negotiator 1 is θ_1^ℓ, type θ_2^k of negotiator 2 would profitably deviate to θ_2^j to get a instead of z, contradicting with strategy-proofness. $\square$

Lemma 11. *Let the mediation rule f be strategy-proof, individually rational, efficient. Then the order $\triangleright$ is transitive. That is, for any triple $a, b, c \in \mathbf{B}$ such that $a \triangleright b$ and $b \triangleright c$, we have $\neg c \triangleright a$.*

Proof. Suppose for a contradiction that there exists $a, b, c \in \mathbf{B}$ such that $a \triangleright b$, $b \triangleright c$ and $c \triangleright a$. There are six possible cases to consider regarding how these three bundles are placed on the main diagonal and symmetric arguments suffice to prove them all. Therefore, we present the proof of one of the cases and the readers can refer to figure 6 for that:

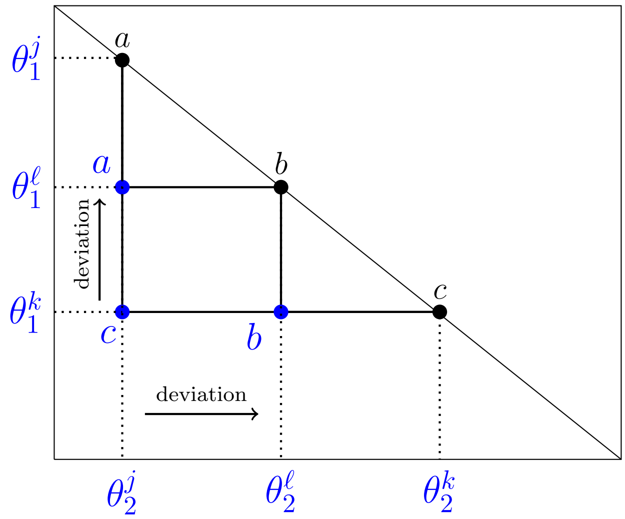


Figure 7

Suppose, without loss of generality, that a appears above bundle b and b appears above bundle c on the main diagonal. Therefore, negotiator 2 prefers a to b and b to c, and type θ_2^j finds all three bundles acceptable. Moreover, $a \rhd b$, $b \rhd c$ and $c \rhd a$ implies $f_{\ell,j} = a$, $f_{k,j} = c$ and $f_{k,\ell} = b$. Given that player 1 is of type θ_1^k, θ_2^j would profitably deviate to type θ_2^ℓ because b is more preferred than c, contradicting with strategy-proofness.

□

Lemma 12. *Let the mediation rule f be strategy-proof, individually rational, efficient and $f_{\ell,j} = a \in \mathbf{B}$ for some $\ell, j \in [0,1]$ with $\ell > j$. Then, $a \rhd b$ for all $b \in \mathbf{B}_{\ell j}$ with $b \neq a$.*

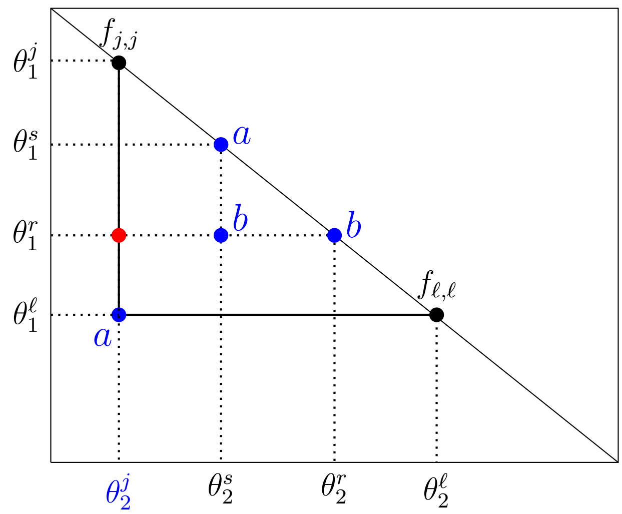


Figure 8

Proof. Suppose for a contradiction that there exists some $b \in \mathbf{B}_{\ell j}$ with $b \neq a$ such that $b \rhd a$. Consider the case where the bundle a is located above the bundle b on the main diagonal. Symmetric arguments will yield a contradiction when a is located below the bundle b on the main diagonal. Suppose that $f_{s,s} = a$ and $f_{r,r} = b$, and so $f_{r,s} = b$.

Strategy-proofness and individual rationality imply that $f_{r,j} = a$: This is true because if $f_{r,j} \in \mathbf{B}_{sr} \setminus \{a\}$, then type θ_1^ℓ would profitable deviate to θ_1^r, and if $f_{r,j} \in \mathbf{B}_{js} \setminus \{a\}$, then θ_1^r would deviate to θ_1^ℓ, all of which contradict with strategy-proofness. With a similar reasoning, we must have $f_{r,s} = a$ given that $f_{r,j} = a$, which contradicts with $a \neq b$: This is true because when $f_{r,s} \in \mathbf{B}_{rs} \setminus \{a\}$, then type θ_2^s would deviate to θ_2^j, contradicting with strategy-proofness. Thus, $a \rhd b$ for all $b \in \mathbf{B}_{\ell j}$ with $b \neq a$.

□

The last lemma proves that a strategy-proof, efficient and individually rational mediation rule picks the maximal element of $\rhd$ on $\mathbf{B}_{\ell j}$ for all $0 \leq \ell, j \leq 1$ with $\ell \geq j$. Namely, $f_{\ell,j} = \boldsymbol{max}_{\mathbf{B}_{\ell j}} \rhd$ for all $\ell \geq j$. By the Sziplrajn's extension theorem (Sziplrajn 1930), one can extend $\rhd$ to a complete order. This extension will clearly preserve the maximal elements in every compact subset $\mathbf{B}_{\ell j}$ because the maximal elements in every set $\mathbf{B}_{\ell j}$ already has a complete relation with all the elements in that set. Finally, Theorem 1 in Tian and Zhoub (1995) proves that quasi upper-semicontinuity is both necessary and sufficient for $\rhd$ to attain its maximum on all compact subsets $\mathbf{B}_{\ell j}$, which completes the proof.

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