

Dynamic Matching Markets and the Deferred Acceptance Mechanism

John Kennes Daniel Monte Norovsambuu Tumennasan*

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Abstract

Many matching markets are dynamic, with one side's priorities often depending on previous allocations. This creates opportunities for manipulations that do not exist in static matching problems. The school-choice problem, for example, exhibits dynamic features: students move considerably across schools and incumbent students and their siblings are often given the highest priorities. In such a dynamic environment, parents can manipulate the period-wise deferred acceptance (DA) mechanism—which has been widely implemented. We analyze the severity of these strategic incentives in dynamic markets. We prove that under a suitable restriction on the schools' priorities, the fraction of agents with an incentive to manipulate the mechanism approaches zero as the market size increases. We also show that this restriction is tight; without it, the mechanism remains manipulable even in large markets. Finally, despite the significant computational complexity inherent in checking each possible strategy for a given player, we provide an algorithm with which to calculate the percentage of markets that can be successfully manipulated. Based on randomly generated data, we find this number to be very low: For markets with 100 schools, this percentage is only 1.58% when each school is endowed with a unit capacity; it drops to 0.04% when the capacity is twenty students per school. Our theoretical results together with our simulations, justify the implementation of the DA mechanism on a period-by-period basis in dynamic markets, and provide further support for its wide use in practice.

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*Kennes: Department of Economics and Business, Aarhus University, Fuglesangs Allé 4, DK- 8210 Aarhus V, Denmark. *E-mail:* jkennes@econ.au.dk. Monte: Sao Paulo School of Economics - FGV, Rua Itapeva, 474- 12° andar, Bela Vista, CEP 01332-000, Sao Paulo, Brazil. *E-mail:* daniel.monte@fgv.br. Tumennasan: Department of Economics and Business, Aarhus University, Fuglesangs Allé 4, DK-8210 Aarhus V, Denmark. *E-mail:* ntumennasan@econ.au.dk. We thank the financial support from the Center for Research in the Foundations of Electronic Markets. Tumennasan thanks the Economics Departments at Harvard University and Boston College for its hospitality and the Danish Council for Independent Research for providing the travel grant (#12-128681). André Diniz and Jan Kjærgaard provided excellent research assistance. We thank the seminar audience at Boston College, Brown University, Louisiana State University, New York University - Abu Dhabi, the 2014 workshop on Advances in Market Design, the 2014 Social Choice and Welfare conference, the International Workshop on Game Theory and Economic Application of the Game Theory Society in honor of Marilda Sotomayor's 70th birthday, and the Sixth Workshop on Matching in Practice in Paris.

1 Introduction

The field of market design has had a great deal of success in helping to redesign assignment markets. The well-known problem of allocating students to public schools illustrates well the importance of the field in practical applications (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2009, 2005).¹ Economists helped organize centralized student-school matching clearinghouses in New York and Boston, which match more than 100,000 students to schools every year.²

The deferred acceptance (DA) mechanism proposed by Gale and Shapley (1962) has played a prominent role in the school choice problem — the (static) problem most closely related to ours. For instance, both Boston and New York adopted the DA mechanism to allocate students to public schools. The primary theoretical justification for this is that the DA results in the student optimal stable matching which is a very desirable property.³ Another key theoretical justification for the prominent role of DA in practice is that the students have no strategic incentives to manipulate the DA mechanism. Consequently, the DA mechanism implements the student optimal stable matching in dominant strategies.

In this paper we study the DA mechanism in dynamic markets. Many important assignment markets are inherently dynamic, as some agents repeatedly participate in the assignment process. In practice, even the school choice problem has dynamic features: families with multiple children participate in the assignment of schools several times.⁴ Even in systems that do not grant siblings priority, there is considerable mobility of students across schools. Schwartz et al. (2009) report that students in New York primary schools move within a single year and across years. Further, the U.S. General Accounting Office reports that “nearly all students change schools at some point before reaching high school.”⁵ This suggests that there is a significant number of students who are reassigned to schools after the initial assignment, which is handled through the DA mechanism. If one considers the subsequent assignments, then the school choice problem can be viewed as a dynamic matching problem.

In addition to the school choice problem, there are other important assignment markets that are dynamic. Examples include the problem of allocating children to public day care centers, in which the same child is assigned to day care centers in consequent periods;⁶ the centralized assignment of teachers to public schools, where the same teacher can participate in the allocation process several times during her teaching career;⁷ and on-campus housing assignments, in which the same student participates multiple times.⁸

¹Other important applications of market design in practice include, for example, the assignment of doctors to hospitals (Roth, 1984; Roth and Peranson, 1999) and the organ exchange programs (Roth et al., 2004, 2005, 2007).

²Recently, Denver and New Orleans have started similar programs.

³Stability in the school choice setting is a notion of fairness (or elimination of justified envy) (Balinski and Sönmez, 1999): if any student finds another school superior to her match under a stable matching then more deserving students must have taken all the available seats at the superior school.

⁴See Dur (2011) for detailed information.

⁵U.S. General Accounting Office (2010).

⁶See Kennes et al. (2014) for detailed information on the current Danish daycare system.

⁷See Pereyra (2013) for detailed information.

⁸See Kurino (2014) for detailed information.

One important feature that is present in many dynamic markets is that the priorities of one or more sides of the market might be history-dependent. For example, in both the day care assignment and the teacher assignment problems a child or teacher cannot be involuntarily displaced at a school to which she is currently assigned. The same is true for the on-campus housing assignment problems. Another such example occurs in the Boston school system, where children enrolled in preschools have a higher priority over other children in that same school. In markets in which the priorities of one side depend on previous allocations, there might be opportunities for manipulations that do not exist in static matching problems.

Given the wide use of the period-by-period DA mechanism, it is natural to ask whether the theoretical justifications of the DA in static settings carry on to dynamic markets. As we discussed earlier, the first justification is stability; Pereyra (2013), Kennes et al. (2014) and Dur (2011) show that under rather weak restrictions on the agents' preferences, the period-by-period deferred acceptance mechanism produces a stable matching. Thus, if the policy makers' goal is to achieve a "fair" allocation, then the use of the DA in dynamic markets is justified. Importantly, though, it turns out that the DA mechanism is manipulable.⁹ The primary reason for this negative result is that history-dependence creates opportunities for manipulations that are absent in static problems. For example, an agent might misreport her preferences in order to affect the priority ranking and get a better allocation in the future.^{10,11}

The results from the dynamic matching literature thus naturally lead us to the main question of this paper: How problematic are the incentives for strategic manipulations of the period-by-period DA mechanism in dynamic markets? We tackle this question through theoretical results and computer simulations. Our benchmark model is a dynamic version of the school choice problem in which successive cohorts of finitely many students, each of them living for two periods, are matched every period to finitely many schools. First, we prove that the incentives for manipulating the DA mechanism in dynamic matching problems vanish as the market size increases. The growth of the market we consider here is the same one considered in Azevedo and Leshno (2013): the set of agents along with the capacities of the schools (not the set of schools) increases. Under this dynamics we identify the conditions for the implementation of this mechanism as the number of participants increases. We show that if each schools' priorities over agents depend on the previous history only through previously enrolled agents – the condition that also guarantees the stability of the DA mechanism (Kennes et al., 2014) –, then the DA mechanism is approximately strategy-proof in large markets. Specifically, in Theorem 5, we show that the fraction of agents who may have incentives to misreport their preferences when all other participants are reporting truthfully

⁹Both Kennes et al. (2014) and Dur (2011) prove an impossibility result: there exists no strategy-proof and stable mechanism. Pereyra (2013) shows that if we restrict the markets to include only seniority-based priorities and time-invariant preferences, then the DA mechanism is strategy-proof.

¹⁰The scope for manipulation generates dissatisfaction and frustration on agents, exemplified by recent report from The Boston Globe (Ebbert (2011)).

¹¹Indeed this incentive for manipulation is present for another celebrated mechanism, the Top-Trading Cycles. In fact, a stronger result is proved in Monte and Tumennasan (2013): the only mechanism that is Pareto efficient, nonbossy and strategy-proof is the sequential dictatorship.

approaches zero as the market size tends to infinity. What our result suggests is that the DA mechanism can be successfully implemented in practice provided that the schools' priorities satisfy the above mentioned condition.

On the other hand, if the schools' priorities depend on the previous allocation in more than simply by assigning the highest priority to incumbent students, then the DA mechanism remains manipulable even in large markets. In fact, this is the case with the assignment system of young children to public day care centers in Denmark, which is one of the practical dynamic matching problems we have in mind. In Denmark, the day care centers adopt a priority structure that violates the condition that we described above. Namely, in the Danish system it is often the case that a day care institution gives high priority to a child who was eligible to participate in the assignment system in the previous period, but has not been allocated to any day care—precisely, this child receives the highest priority than those who attended any day care institution other than the one in question. This rule is denoted “child care guarantee.” If the deferred acceptance mechanism is applied period-by-period in a market with a priority structure that follows the Danish priorities, the system remains manipulable even in a large market.

We then proceed through simulations to show how often agents have an incentive to manipulate the DA mechanism in dynamic matching markets. There is, however, a significant computational complexity involved here. To illustrate, consider a market with 30 schools. In this market, any given agent has a strategy set of more than 10^{32} ($30!$) elements. Thus, checking each possible manipulation is not a feasible task. In static marriage problems it is well-known that to evaluate whether an agent, say a woman, has an incentive to manipulate the men-proposing DA mechanism it suffices to check the truncation strategies – those in which the woman only reduces her acceptable matches without rearranging her preference ranks of the men. It turns out, however, that sometimes agents in dynamic matching markets can manipulate the DA mechanism only by rearranging their preference ranking of the schools. Therefore, truncating or dropping strategies are not sufficient to identify whether the DA mechanism is manipulable in certain markets. At the same time, as we mentioned above, one cannot hope to consider all of an agent's possible preference reports because the number of these reports increases exponentially as the market size increases. We provide an algorithm that drastically decreases the complexity of this problem, making it feasible to check every possible manipulation for any given agent. The first observation we use in the construction of our algorithm is that any two-year old student should reveal her true preferences because the DA mechanism is strategy-proof in static settings. Thus, only one-year-old agents can have an incentive to manipulate the DA mechanism. Let us fix a one year old child and suppose that all the other agents reveal their preferences truthfully. If this one-year-old agent can manipulate the DA successfully by submitting a particular preference report, then all the preference reports that lead to the same matching must also be a beneficial manipulation for the agent. It turns out that in a typical market, there is only a limited number of matchings produced by the DA mechanism when an agent varies her preference reports. We use this result in the construction of our algorithm. Specifically, in each round our algorithm finds that the DA matching corresponds to some report of

a given one year agent and disregards all the preference reports that lead to the same DA matching. Thus, our algorithm runs the same number of rounds as the number of different DA matchings to which the preference reports of the one-year-old agent leads.

Using the algorithm described above, which we believe is of independent interest, we estimate the percentage of the markets in which a given agent can successfully manipulate the DA mechanism based on randomly generated data. In markets with fifty schools, this percentage is 1.05% if each school's capacity is one; it drops to 0.03% if each school's capacity is 20. In markets with 100 schools, the manipulation percentage is 1.58% if each school's capacity is one; it drops to 0.04% when the capacity is 20. Therefore, the manipulation percentage of the DA mechanism quickly converges to 0.

Our theoretical and simulation results therefore justify the use of the period-by-period DA mechanism in practice. For instance, the Boston school district uses a system with sibling priority and assigns the students to schools using the period-by-period DA. Given the schools' capacity and the number of the students, our analysis shows that the manipulability of the DA mechanism in the Boston Public School system is negligible. Similarly, the school districts can handle the assignment of students who move into their district as well as the students who want to switch schools after the initial DA placement through the DA mechanism without fearing strategic manipulations. The same conclusion holds for the assignment of the agents to day care institutions in Denmark.

1.1 Related Literature

Our paper is related both to the literature on dynamic matching as well as the literature on matching in large markets. Kurino (2014), Pereyra (2013), Dur (2011) and Kennes et al. (2014) study the centralized matching when the set of agents evolve in the overlapping generations fashion. Kurino (2014) focuses on the house allocation problem, and he shows that seniority based top trading cycles mechanism is dynamically efficient, acceptable and strategy-proof under time-invariant preferences. Pereyra (2013) shows that the DA mechanism is stable and strategy-proof under seniority based priorities and time-invariant preferences. Because the older generation gets a higher priority than the younger generation and the agents have time-invariant preferences in both studies, their corresponding mechanisms effectively consider the older generation first and then the younger generation in each period. This is the main reason why Kurino (2014) and Pereyra (2013) obtain positive results. Dur (2011) models the school choice problem as a dynamic problem taking the sibling priorities into account. On the other hand, Kennes et al. (2014) consider the problem of allocating children to day care centers where each child attends day care centers in multiple periods and participating children evolve in the overlapping generations fashion. Both Dur (2011) and Kennes et al. (2014) show that (i) the DA mechanism results in a stable matching and (ii) no mechanism is both stable and strategy-proof. The key reason for their impossibility results is that they consider a much broader class of schools' priorities than the seniority based ones, due to the applications considered in these papers. Bloch and Cantala (2013)'s dynamic matching problem is similar to ours, but they focus on the long-run properties of different assignment rules. Ünver

(2010) studies the kidney exchange problem considering a dynamic environment in which the pool of agents evolves over time.

In this paper we use the concept of stability for dynamic environments that we introduced in Kennes et al. (2014). Other authors have worked with similar concepts before us; Kurino (2009a), Kurino (2009b), and Damiano and Lam (2005) had a different concept than ours, assuming farsighted agents. More recently, Doval (2014) introduced a dynamic stability concept, but her model differs from ours in that agents arrive over time and are matched at most once. Kadam and Kotowski (2014) study a repeated two-sided market and introduce different stability notions, more suitable to their environment, in which the classes of preferences and priorities are different from ours.

Compte and Jehiel (2008) introduced a model in which agents are reallocated to positions with the restriction that in their new position they are not worse off than they were initially. Thus, our problem relates to theirs in the sense that we assume property-rights in the priority system of schools; students allocated to schools cannot be involuntarily placed in worse schools in subsequent periods.

Our paper is also related to the literature on large matching markets, for which there is a broad recent literature.¹² Mostly, it relates to Roth and Peranson (1999), Immorlica and Mahdian (2005), Kojima and Pathak (2009) and in particular, to Azevedo and Leshno (2013). Roth and Peranson (1999) stipulate that the percentage of participants who can successfully manipulate the DA mechanism converges to 0 as the market size increases based on a series of simulations on the National Residence Matching Program data and on randomly generated data. Immorlica and Mahdian (2005) and Kojima and Pathak (2009) consider one-to-one and many-to-one settings respectively and show that the incentives to manipulate the DA mechanism vanishes as the market size increases in their respective settings. In both Immorlica and Mahdian (2005) and Kojima and Pathak (2009), the both sides of the market grow whereas in our study one side of the market, namely agents, increases along with the capacities of the schools. In static matching settings Che and Kojima (2010), Abdulkadiroğlu et al. (2014), Che and Tercieux (2014), Azevedo and Hatfield (2013) and Azevedo and Leshno (2013) consider the dynamics of the market size growth we study here: a large number of agents matched to a finite number of objects. Che and Kojima (2010) show that the probabilistic serial dictatorship mechanism becomes strategy-proof as the number of agents along with the copies of the objects tend to infinity because this mechanism is equivalent to the random serial dictatorship mechanism. Che and Tercieux (2014) study efficiency and stability in large (static) matching markets and focus on the payoffs achieved by the agents in a large class of mechanisms—not just the DA. Azevedo and Leshno (2013) consider the convergence of stable matchings in many-to-one matching settings as the market size increases, and they show that in a wide class of markets the stable matchings converge to a matching which is a unique stable matching

¹²See Roth and Peranson (1999); Immorlica and Mahdian (2005); Kojima and Pathak (2009); Manea (2009); Che and Kojima (2010); Kojima and Manea (2010); Kojima et al. (2013); Liu and Pycia (2012); Azevedo and Leshno (2013); Che and Tercieux (2014); Abdulkadiroğlu et al. (2014); Azevedo and Hatfield (2013); and Azevedo and Budish (2013).

in the continuum economy. We use some of the Azevedo and Leshno (2013)’s results extensively, but our paper differs from Azevedo and Leshno (2013)’s in two major aspects: (i) our focus is the manipulation of the DA while theirs is stability in large or continuum economies, and (ii) we study dynamic environments while Azevedo and Leshno (2013) concentrate in a static setting.

The paper is organized as follows: in Section 2, we provide the model and the main definitions. In Section 3 we describe a version of the deferred acceptance mechanism, from Kennes et al. (2014). In Section 4 we examine the main properties of the mechanism in small economies. In Section 5 we provide our algorithm to check whether an agent can manipulate the DA mechanism, and we present our simulation results on the manipulability of the DA. Section 6 contains the results for an economy with a continuum of agents. In Section 7 we prove our main convergence result. The longer proofs are in the Appendix.

2 Model

2.1 Setup

Time t is discrete and $t = 1, \dots, \infty$. There is a finite number of infinitely lived schools.¹³ Let $S = \{h, s_1, \dots, s_m\}$ be the set of schools as well as the option of staying home, h . Let $\mathbf{r} = (r^s)_{s \in S}$ be the vector of capacities, with $r^s \in \mathbb{N}$. We assume that each school other than home has a finite capacity, $r^s < \infty$, for all $s \neq h$ whereas home does not have a capacity constraint, that is, $r^h = \infty$.

We assume that each agent can attend school when she is one and two years old.¹⁴ If an agent attends schools s and s' when she is 1 and 2 respectively, then we write (s, s') to denote the allocation of this agent.

An agent i is one year old in period t_i . In addition, this agent has a strict preference relation \succ_i over the set of possible pairs of schools, and is initially endowed with a priority score vector, $\mathbf{x}_i = (x_i^s)_{s \in S} \equiv [0, 1)^{m+1}$. It is convenient to think that each agent i is a triplet $(t_i, \succ_i, \mathbf{x}_i)$ and to write that $i = (t_i, \succ_i, \mathbf{x}_i)$.

At period $t \geq 1$, a finite set of one year old agents I_t arrives, i.e., $i \in I_t$ if and only if $t_i = t$. We use the notation I_0 to denote the set of the agents who are two in period 1. Consequently, at any period $t \geq 1$ the set of school-age agents is $I_{t-1} \cup I_t$. As time passes the set of school-age agents evolves in the “overlapping generations” (OLG) fashion. Let $I = (I_t)_{t=0}^\infty$. A finite economy $E = (I, \mathbf{r})$ specifies a finite set of agents per cohort and a vector of capacities.

We now define the matching in our setting.

Definition 1 (Matching). *A period-0 matching μ_0 is a correspondence $\mu_0 : I_0 \cup S \rightarrow I_0 \cup S$ such*

¹³Equivalently, one could think of daycare centers instead of schools, as in Kennes et al. (2014)’s prototypical application.

¹⁴In the school choice setting with siblings, this assumption is equivalent to the one in which each family has two children. Incorporating families with one child into the model is straightforward (Dur, 2011): in each period there is a set of agents who participate in the allocation process only once in the period they are one and who have preferences over schools. Now from the school choice literature, we know that these agents will not have any justified envy or strategic manipulation if the DA mechanism is used.

that

1. For all $i \in I_0$, $\mu_0(i) = \{h\}$.
2. $\mu_0(h) = I_0$ and $\mu_0(s) = \emptyset$, for all $s \neq h$

A period- t matching at any $t \geq 1$, μ_t , is a correspondence $\mu_t : I_{t-1} \cup I_t \cup S \rightarrow I_{t-1} \cup I_t \cup S$ such that

1. For all $i \in I_{t-1} \cup I_t$, $|\mu_t(i)| = 1$ and $\mu_t(i) \subset S$
2. For all $s \in S$, $|\mu_t(s)| \leq r^s$ and $\mu_t(s) \subset I_{t-1} \cup I_t$ ¹⁵
3. For all $i \in I_{t-1} \cup I_t$, $i \in \mu_t(s)$ iff $s \in \mu_t(i)$.

A matching μ is a collection of period matchings: $\mu = (\mu_t)_{t=1}^\infty$.

We slightly abuse the notation by using the notation $\mu_t(i)$ to denote the school to which agent i is matched under μ_t . We use the notation $\mu(i)$ to denote the pair of schools that i is matched with under matching μ : $\mu(i) = (\mu_{t_i}(i), \mu_{t_i+1}(i))$. Let \mathcal{M}_t be the set of period t matchings.

From the definition above, observe that in period 0, every agent stays home, i.e., the schools start their operation at period 1. As a consequence of this assumption, all matchings we consider have a common period-0 matching in which all school-age agents are matched with h .¹⁶

Agents' Preferences

We already noted that each agent i has strict preferences, \succ_i . We write $(s, s') \succeq_i (\bar{s}, \bar{s}')$ if either $(s, s') \succ_i (\bar{s}, \bar{s}')$ or $(s, s') = (\bar{s}, \bar{s}')$.

We will impose some restrictions on agents' preferences. First we assume that each agent has an underlying ranking over schools (not the pairs of schools) which stays stable over time.¹⁷ Consequently, if a school s is superior than another school s' , then it must be that $(s, s) \succ_i (s', s')$. In addition, we assume that there is no complementarity from attending two different schools but there could be from attending the same school for two periods. Specifically, attending an inferior school s' in one period and a different school s'' in the other eligible period is always worse than attending a superior school s and school s'' . On the other hand, attending s' for two periods maybe better than attending the superior school s in one period and s' in the other eligible period. This latter scenario is could be due to switching costs. These assumptions are collected in the following assumption, which we will maintain throughout the paper.

¹⁵It can be that $\mu_t(s) = \emptyset$ for some school s . In such cases, no agent attends s at period t .

¹⁶We can relax this restriction so that a period-0 matching is defined similarly to the other period matchings. Given that we interpret period 0 as the period that occurred right before the start of our model, period-0 matching cannot be altered. Thus, all matchings must have a common period-0 matching which is one of the primitives of the model. With the modified definition of a period-0 matching all the results except those in section 7 go through. In Remark 1 we will present an additional assumption that guarantees the validity of results in Section 7.

¹⁷In the school choice with siblings setting this means that two children of the same household have the same preferences of schools. Although one can think of cases in which this assumption is violated, we believe that our assumption is valid for majority of cases, especially for the elementary or middle school students.

Assumption 1 (Rankability). *If $(s, s) \succ_i (s', s')$ for some i, s and s' , then $(s, s'') \succ_i (s', s'')$ and $(s'', s) \succ_i (s'', s')$ for any $s'' \neq s'$.*

Let \mathcal{R} be the set of preferences satisfying Assumption 1.

Now let us define a stronger version of the rankability assumption which rules out the possibility that attending an inferior school for two periods is better than attending this school for one period and a superior school in the other eligible period.

Definition 2 (Strong Rankability). *If $(s, s) \succ_i (s', s')$ for some i, s and s' , then $(s, s'') \succ_i (s', s'')$ and $(s'', s) \succ_i (s'', s')$ for any s'' .*

The strong rankability assumption means that the switching costs are relatively small. We here remark that the sole purpose of the strong rankability assumption is to simplify the presentation of some of our examples, i.e., none of our results rely on this stronger assumption.

Before we move on, consider period $t \geq 1$, and suppose that the period matchings up till this period is assigned. Let us now assume that the agents who are eligible to attend school in this period are asked to rank the schools (not pairs of schools). Given the motivations for the rankability assumption, perhaps any one year old agent i would rank s over s' if and only if s is superior to s' , i.e., if and only if $(s, s) \succ_i (s', s')$. A two year old, on the other hand, would rank s over s' if and only if $(\mu_t(i), s) \succ_i (\mu_t(i), s')$ given that she knows her match in the previous period. This is the motivation behind the concept of isolated preferences considered by Kennes et al. (2014).

Definition 3 (Isolated Preference Relation). *For any given period $t \geq 1$, and for a given matching μ_{t-1} , the isolated preference relation of period t , $P_i(\mu_{t-1})$ is a binary relation satisfying*

1. *For $\forall i \in I_t : sP_i(\mu_{t-1})s'$ if and only if $(s, s) \succ_i (s', s')$ for any $s \neq s' \in S$*
2. *For $\forall i \in I_{t-1} : sP_i(\mu_{t-1})s'$ if and only if $(\mu_{t-1}(i), s) \succ_i (\mu_{t-1}(i), s')$ for any $s \neq s' \in S$.*

We here note that in any period the isolated preferences of 1-year old agents do not depend on the preceding period's matching. Let $P(\mu_{t-1})$ be the collection of isolated preferences for the school-age agents in period t , i.e., $P(\mu_{t-1}) \equiv (P_i(\mu_{t-1}))_{i \in I_{t-1} \cup I_t}$. In addition, the notation \mathcal{P} denotes the set of all possible isolated preferences. We will usually write P_i instead of $P_i(\mu_{t-1})$ as long as doing so does not create confusion.

We note here that if agent's preferences are strongly rankable, then her isolated preferences will be the same regardless of her age or previous period's matching.

Schools' Priorities and History Dependence

We previously mentioned that each agent i is endowed with a priority score vector \mathbf{x}_i . This priority score vector will be used to determine the agent's priority in each school in the period that this agent is one-year old, i.e., in period t_i . It is fixed at the agent's birth year, but may change in the following period when the schools' priorities are history-dependent, which we will assume.

If an agent i has a priority score vector \mathbf{x}_i and an agent j , born in the same period has a priority score vector given by \mathbf{x}_j , with $x_i^s > x_j^s$, for some $s \in S$, we have that agent i has a higher priority in period t_i than agent j at school s .

Assumption 2 (Strict Priorities). *For any two agents i, j in $I_{t-1} \cup I_t$, $x_i^s \neq x_j^s$ for all $s \in S$.*

Given the dynamic nature of our problem, in our model we will consider the case in which the priority score of agent i at period $t_i + 1$ depends on the previous period's matching. Conceptually, we do not impose any restriction on how this history-dependence occurs, that is, we imagine that there might be dynamic matching problems in which the priority vector of an agent varies across time in a variety of different ways. However, in our model we will consider only two specific channels through which the priority vector of an agent may change over time. This is motivated by natural applications of dynamic matching problems, such as the centralized assignment of young children to public day care centers, the assignment of teachers to public schools and when there is priority for incumbent students and their siblings in the school choice problem.

The two instances in which the priority vector of an agent might change from one period to another are: 1) the schools give the highest priorities to their currently enrolled agents; and 2) schools give priority to agents that were not enrolled in any school in the previous period. The first channel is motivated by the concept of property-rights: in our example, this would imply assigning high priorities to previously allocated students. This feature is present in many real-life applications. In particular, this is the case in the Danish day care system. It is also the case in other assignment problems, for example in the assignment of teachers to public schools in countries as diverse as France, Brazil, and Mexico. Given the importance of this restriction on many different systems and on its natural appeal, i.e. agents will not be forced out of a school, we will maintain this assumption throughout our paper.

To incorporate the history-dependence of priorities in our model, we define the *priority score function* of each agent i at school some school s as a mapping $X_i^s : \mathcal{M}_{t_i-1} \cup \mathcal{M}_{t_i} \rightarrow [0, 2]$ such that $X_i^s(\mu_{t_i-1}) = x_i^s$ for all $\mu_{t_i-1} \in \mathcal{M}_{t_i-1}$. This means that if a period- t matching was $\mu_t \in \mathcal{M}_t$, then at period $t + 1$ the priority score of a school-age agent i at school s is $X_i^s(\mu_t)$.¹⁸ If i was born in period $t + 1$, then her priority score at school s must be $X_i^s(\mu_t) = x_i^s$, which is exogenously determined.

For a given matching $\mu_t \in \mathcal{M}_t$, we denote the priority score vector of s at period $t + 1$ by $X^s(\mu_t) \equiv (X_i^s(\mu_t))_{i \in I_t \cup I_{t+1}}$ and school-age agent i 's priority scores at all schools by $X_i(\mu_t) \equiv (X_i^s(\mu_t))_{s \in S}$. We will maintain the following assumption throughout the paper.

Assumption 3 (Independence of Past Attendances (IPA)). *Each agent's priority score function at any school s satisfies that:*

$$X_i^s(\mu_{t_i}) = \begin{cases} 1 & \text{if } i \in \mu_{t_i}(s) \\ x_i^s & \text{otherwise} \end{cases}$$

¹⁸Moving forward it is convenient to have one notation that expresses the priority scores of both one- and two-year-old agents.

for all $\mu_{t_i} \in \mathcal{M}_{t_i}$.

This assumption states that an agent who is matched to a school s when she is one will have the highest priority score at school s when she is two. In addition, the agent's priority score at any other school remains the same unless she was matched to that school at the age of 1. Here, observe that the attendees of any school s at some period t will have the same priority score of 1 at the school in the following period. This assumption, as we will see later, does not cause any problem to run the version of the deferred acceptance algorithm used in this paper—note that given assumption 2 we will not have the problem that there are more students with the same score at a school than the school's capacity.

In the current Danish day care assignment system *IPA* is not satisfied: if an older child who has not attended any day care previously asks for a guaranteed spot then at some day care she is given a priority over all the children who have attended some day care previously as well as some younger children who are participating in the assignment process for the first time.¹⁹ The current rule does not spell out which day care this older child gets a priority.²⁰ Due to this incompleteness of the rule, we cannot accurately define the current Danish priority system. However, given the importance of the day care assignment system in practice, we would like to examine the current Danish priority system closely when we study the incentives to manipulate the deferred acceptance mechanism. Consequently, in section 6, in particular in theorem 4 we will consider the case in which assumption 3 is violated in a specific way: each school assigns a higher priority to two-year-old children who have not attended any school in the previous period over one-year-old children and two-year-old children who previously attended a school other than the one in question.²¹ That is, we maintain the property-right feature of our *IPA* assumption, but also add the feature that children who have not attended any school have higher priority over other children who attended some school.

Assumption 3a (Failure of *IPA*: Danish Priorities). *A priority scoring system is Danish if each agent's priority score function at each school s satisfies the following condition condition:*

$$X_i^s(\mu_{t_i}) = \begin{cases} 2 & \text{if } i \in \mu_{t_i}(s) \\ 1 + x_i^s & \text{if } i \in \mu_{t_i}(h) \\ x_i^s & \text{otherwise} \end{cases}$$

for all $\mu_{t_i} \in \mathcal{M}_{t_i}$.

In the Danish priority scoring system an agent who stays at home when she is young will have a priority score of $1 + x_i^s$ at school s in the following period. Consequently, by staying home at age 1, an agent jumps ahead of almost all agents (except the school's previous period's attendees) in the priority ranking of any school at age 2. However, observe here that the relative rankings of those who stay home when they are 1 do not change.

¹⁹Children who have special needs (due to disability or due to a foreign language spoken at home) or who have siblings at a specific daycare always have higher priority over the children who ask for a guaranteed spot

²⁰It looks like the officials in charge of the assignments decide this.

²¹This assumption is stronger than what is actually done in practice, however, it illustrates the issue well.

Threshold Scores

For a given matching μ , let the period- t threshold score of school s corresponding to μ be p_t^s such that

$$p_t^s = \begin{cases} 0 & \text{if } |\mu_t(s)| < r^s \\ \inf_{i \in \mu_t(s)} X_i^s(\mu_{t-1}) & \text{otherwise} \end{cases}$$

Observe here that the threshold score of h corresponding to any matching is always 0 because h does not have any capacity restriction. We use the following notations: $\mathbf{p}_t = (p_t^s)_{s \in S}$ and $\mathbf{p} = (\mathbf{p}_t)_{t=1}^\infty$.

2.2 Properties of a Matching

As we have discussed in the introduction, stability is a much desired property in the school choice problem. Kennes et al. (2014) proposed a stability concept for the day care assignment problem,²² and in order to avoid new notations, we here use an equivalent definition when the schools' priorities satisfy IPA.²³

Definition 4 (Stability). *A matching μ satisfies stability if at any period $t \geq 1$, there does not exist a school-agent pair (s, i) such that (1) and (2) below hold at the same time*

1. (a) $(s, \mu_{t+1}(i)) \succ_i (\mu_t(i), \mu_{t+1}(i))$, or
 (b) $(s, s) \succ_i (\mu_t(i), \mu_{t+1}(i))$, or
 (c) $(\mu_{t-1}(i), s) \succ_i (\mu_{t-1}(i), \mu_t(i))$,
2. $|\mu_t(s)| < r_s$ or/and $X_i^s(\mu_{t-1}) > X_j^s(\mu_{t-1})$ for some $j \in \mu_t(s)$.

At any stable matching each agent should not improve if she is transferred for one or two periods to any school that has an agent who has a lower priority than the original agent. Thus, no agent has a justified envy of another agent at a stable matching.

Kennes et al. (2014) show that a stable matching exists as long as the schools' priorities satisfy IPA. Moreover, the *Deferred Acceptance* mechanism, which we will define formally later, results in a stable matching. We note here that the existence of stable matchings is not guaranteed if the preferences do not satisfy IPA.²⁴

2.3 Mechanism

A mechanism (for finite economies) is a systematic process that assigns a matching for each finite economy. We use the notation φ to denote a typical mechanism for finite economies. Let $\varphi_i(E)$ be

²²We were not the first ones to introduce the notion of stability to a dynamic context. Other definitions prior to ours include Kurino (2009a)'s notion of dynamic pairwise-stability, and Damiano and Lam (2005)'s self-sustaining stability. Differently from our stability notion, in these authors' notions, it is assumed that agents are farsighted. A more recent paper by Kadam and Kotowski (2014) introduces the notions of ex-ante and dynamic stability– which are complementary to our notions– in a dynamic two-sided matching markets.

²³See Lemma 2 of Kennes et al. (2014)

²⁴See Kennes et al. (2014).

the pair of schools to which agent i is matched under φ . We now introduce two desirable properties of a mechanism.

Definition 5. *A mechanism is stable if it associates each finite economy E with a stable matching in E .*

For each mechanism, there is an associated preference revelation game. If no agent has incentives to misrepresent her preferences in this game, then we say the mechanism is strategy-proof. Below we state the formal definition.

Definition 6 (Strategy-Proofness). *We say that a mechanism φ is manipulable (individually) at a finite economy E if there exists an economy $E' = (I', \mathbf{r})$, and an agent $i \in I$, such that*

1. E' differs from E only in agent i 's preference ordering (i.e., $\succ_i \neq \succ'_i$ and $\succ_j = \succ'_j$, for all $j \neq i$) and
2. $\varphi_i(E') \succ_i \varphi_i(E)$.

A mechanism φ is strategy-proof if it is not manipulable at any finite economy.

3 Deferred Acceptance Mechanism using Isolated Preferences

We argued that the isolated preferences represents the agents' period preferences over schools when we considered the agents' preferences. Consequently, one of the most natural mechanisms is the period-by-period Gale and Shapley deferred acceptance mechanism that utilizes the isolated preferences. Kennes et al. (2014) consider this mechanism and denote it by *Deferred Acceptance Mechanism using Isolated Preferences* (DA-IP). In this mechanism the school-age agents at any given period report their isolated preferences over schools knowing their previous period's matchings. Formally, the DA-IP mechanism associates each economy with the matching that is the result of the DA-IP algorithm which we define below.

Fix a finite economy E . Because this paper revolves around the DA-IP mechanism, we reserve the notation η for the matching that is the result of the DA-IP algorithm. Recall that for all matchings, every agent stays home in period 0. Thus, every agent is assigned h at η_0 . The DA-IP algorithm determines period-1 DA-IP matching η_1 using η_0 . Once the period-1 DA-IP matching is determined, the algorithm uses this matching to determine the period-2 DA-IP matching η_2 , and it does so for every subsequent period. The period-1 DA-IP matching is found by running the following algorithm in finite rounds (in essence, the well-known deferred acceptance algorithm).

Period-1 assignment:

Set the isolated preferences for each school-age agent i in this period to $P_i(\eta_0)$. In addition, set the priority score vector of each school s in this period to $X^s(\eta_0)$.

Round 1: Each school-age agent of period 1 applies to her most preferred school according to her isolated preferences ($P_i(\eta_0)$ in this case). Each school s “holds” the r_s applicants with the highest priority score (according to $X^s(\eta_0)$ in this case) and rejects all others.

In general, at:

Round k : Each agent whose application was rejected in the previous round applies to her most preferred school (according to her isolated preferences, $P_i(\eta_0)$ in this case) that has not rejected her. Each school s considers the pool of applicants composed of the new applicants and the agents whom s has been holding from the previous round. Each school s then “holds” the r_s agents in the pool who have the highest priority score (according to $X^s(\eta_0)$ in this case) and rejects all others.

The algorithm terminates when no proposal is rejected and each agent is assigned her final tentative assignment. This final matching is η_1 .

Period-2 assignment:

Set the isolated preferences of each school-age i in this period to $P_i(\eta_1)$ and the priority score vector of each school s in this period to $X^s(\eta_1)$. Now using the algorithm described above, we can find the period-2 DA-IP matching η_2 .

Period- t assignment:

Set the isolated preferences of each school-age i in this period to $P_i(\eta_{t-1})$ and the priority score vector of each school s in this period to $X^s(\eta_{t-1})$. Now using the algorithm described above, we can find the period- t DA-IP matching η_t .

Kennes et al. (2014) prove the following result:²⁵

Theorem 1 (DA-IP is Stable). *If the schools’ priorities satisfy IPA, then the DA-IP mechanism is stable.*

4 Manipulation in Small Economies

It is well known that in static settings, the student proposing DA mechanism is strategy-proof. In contrast, Kennes et al. (2014) show that in dynamic environments the DA-IP mechanism is not strategy-proof,²⁶ i.e., in some small economies an agent finds it profitable to misrepresent her preferences (when everyone else reports her preferences truthfully).

Theorem 2 (DA-IP is Manipulable). *The DA-IP mechanism is not strategy proof.*

Below we present an example that proves the theorem above because it will be used in our further analysis. This example is a simplified version of the economy that Kennes et al. (2014) use in their proof of the impossibility result.

²⁵Precisely, see Theorem 1 of Kennes et al. (2014).

²⁶Their result is even stronger: no strongly stable and strategy proof mechanism exists.

Example 1. Consider the following economy E with 3 schools $\{s, s_1, s_2\}$. Each school has a capacity of one agent. Suppose $I_0 = \{i_0\}$, $I_1 = \{i_1, i_2\}$, $I_2 = \{i_3\}$ and $I_\tau = \emptyset$, for all $\tau \geq 3$. In addition, suppose that the priority score vectors are such that:

$$\begin{aligned} x_{i_0}^s &> x_{i_3}^s > x_{i_1}^s > x_{i_2}^s \\ x_{i_0}^{s_1} &> x_{i_1}^{s_1} > x_{i_2}^{s_1} > x_{i_3}^{s_1} \\ x_{i_0}^{s_2} &> x_{i_1}^{s_2} > x_{i_3}^{s_2} > x_{i_2}^{s_2} \end{aligned}$$

Suppose that each agent's preferences satisfy strong rankability, which means that the agents' isolated preferences are independent of the past matchings. The agents' isolated preferences are given as follows:

$$\begin{aligned} i_0 &: s \\ i_1 &: s \quad s_1 \quad s_2 \\ i_2 &: s_2 \quad s_1 \quad s \\ i_3 &: s_2 \quad s \quad s_1 \end{aligned}$$

In addition, suppose $(s_2, s) \succ_{i_1} (s_1, s_1)$. We denote the DA-IP matching in economy E by η . The DA-IP algorithm in period 1 yields the following matching:

$$\eta_1 = \begin{pmatrix} s & s_1 & s_2 \\ i_0 & i_1 & i_2 \end{pmatrix}.$$

In period 2, the priority scores are updated such that:

$$\begin{aligned} X_{i_3}^s(\eta_1) &> X_{i_1}^s(\eta_1) > X_{i_2}^s(\eta_1) \\ X_{i_1}^{s_1}(\eta_1) &> X_{i_2}^{s_1}(\eta_1) > X_{i_3}^{s_1}(\eta_1) \\ X_{i_2}^{s_2}(\eta_1) &> X_{i_1}^{s_2}(\eta_1) > X_{i_3}^{s_2}(\eta_1) \end{aligned}$$

The DA-IP algorithm in period 2

$$\eta_2 = \begin{pmatrix} s & s_1 & s_2 \\ i_3 & i_1 & i_2 \end{pmatrix}.$$

In particular, note that agent i_1 's final allocation is

$$\eta(i_1) = (s_1, s_1).$$

Consider the following finite economy E' , which differs from E only in agent i_1 's preference profile, which is still strongly rankable and given by:

$$(s, s) \succ_{i_1} (s_2, s_2) \succ_{i_1} (s_1, s_1).$$

In economy E' , agent i_1 ranks the schools as follows according to her isolated preferences:

$$i_1 : s \quad s_2 \quad s_1.$$

Denote by η' the matching resulted from the DA-IP algorithm applied to economy E' . Period 1 matching is

$$\eta'_1 = \begin{pmatrix} s & s_1 & s_2 \\ i_0 & i_2 & i_1 \end{pmatrix}.$$

In period 2, the updated priority scores are:

$$\begin{aligned} X_{i_3}^s(\eta'_1) &> X_{i_1}^s(\eta'_1) > X_{i_2}^s(\eta'_1) \\ X_{i_2}^{s_1}(\eta'_1) &> X_{i_1}^{s_1}(\eta'_1) > X_{i_3}^{s_1}(\eta'_1) \\ X_{i_1}^{s_2}(\eta'_1) &> X_{i_3}^{s_2}(\eta'_1) > X_{i_2}^{s_2}(\eta'_1) \end{aligned}$$

Based on this updated priority scores and the isolated preferences, the DA-IP algorithm yields the following matching at period 2:

$$\eta'_2 = \begin{pmatrix} s & s_1 & s_2 \\ i_1 & i_2 & i_3 \end{pmatrix}.$$

In particular, note that the allocation of agent i_1 in economy E' is (s_2, s) .

This shows that the DA-IP mechanism is manipulable at economy E .

Observe from the example above that an agent gets strictly worse when she is one but gets better when she is two at her successful manipulation. However, it turns out that a manipulating agent does not have to sacrifice her first period assignment. The following example demonstrates this point.

Example 2. Consider the following economy E with five schools s_1, s_2, s_3, s_4 and s_5 , and six agents, i_1, i_2, i_3, i_4, i_5 and i_6 . Each school has a capacity of one agent and suppose that $I_0 = \{i_1, i_2\}$, $I_1 = \{i_3, i_4, i_5\}$, $I_2 = \{i_6\}$.

Each agent's preferences are strongly rankable, and the agents' isolated preferences are as follows:

$$\begin{aligned} i_1 : & s_1 \\ i_2 : & s_2 \\ i_3 : & s_1 \quad s_5 \quad s_3 \quad s_4 \quad s_2 \\ i_4 : & s_4 \quad s_2 \quad s_3 \quad s_5 \quad s_1 \\ i_5 : & s_3 \quad s_1 \quad s_4 \quad s_5 \quad s_2 \\ i_6 : & s_4 \quad s_1 \quad s_3 \quad s_5 \quad s_2 \end{aligned}$$

The priority scores of the agents are follows:

$$\begin{aligned}
x_{i_1}^{s_1} &> x_{i_6}^{s_1} > x_{i_3}^{s_1} > x_{i_4}^{s_1} > x_{i_5}^{s_1} > x_{i_2}^{s_1} \\
x_{i_2}^{s_2} &> x_{i_4}^{s_2} > x_{i_1}^{s_2} > x_{i_3}^{s_2} > x_{i_5}^{s_2} > x_{i_6}^{s_2} \\
x_{i_6}^{s_3} &> x_{i_4}^{s_3} > x_{i_3}^{s_3} > x_{i_5}^{s_3} > x_{i_1}^{s_3} > x_{i_2}^{s_3} \\
x_{i_6}^{s_4} &> x_{i_5}^{s_4} > x_{i_3}^{s_4} > x_{i_4}^{s_4} > x_{i_1}^{s_4} > x_{i_2}^{s_4} \\
x_{i_3}^{s_5} &> x_{i_5}^{s_5} > x_{i_4}^{s_5} > x_{i_2}^{s_5} > x_{i_1}^{s_5} > x_{i_6}^{s_5}
\end{aligned}$$

Let E' be an economy which differs from E only in agent i_3 's preferences, which are still strongly rankable and given by:

$$i_3 : s_1 \ s_3 \ s_5.$$

The DA-IP matchings in economies E and E' are given

$$\begin{array}{c}
\text{Economy } E \\
\text{Economy } E'
\end{array}
\left(\begin{array}{cc} \text{Period 1} & \text{Period 2} \end{array} \right)
\begin{array}{c}
\left(\begin{array}{cc} s_1 & s_2 & s_3 & s_4 & s_5 \\ i_1 & i_2 & i_5 & i_4 & i_3 \end{array} \right) \\
\left(\begin{array}{cc} s_1 & s_2 & s_3 & s_4 & s_5 \\ i_1 & i_2 & i_4 & i_5 & i_3 \end{array} \right)
\end{array}
\left(\begin{array}{cc} s_1 & s_3 & s_4 & s_5 \\ i_6 & i_5 & i_4 & i_3 \end{array} \right)
\left(\begin{array}{cc} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_5 & i_6 \end{array} \right)$$

Thus, agent i_3 has an incentive to manipulate the DA-IP mechanism at economy E . Interestingly, agent i_3 still gets to match with school s_5 in period 1 but gets to match with a better school, s_1 in period 2.

In this example, agent i_3 manipulates the DA-IP successfully by ranking s_3 ahead of s_5 in her preference report of period 1, and there is no other preference report that improves i_3 over truth telling. This means that agents must alter the relative ranking of the schools in order to successfully manipulate the DA-IP mechanism.

Observe from the examples above that at her successful manipulation an agent gets worse when she is one but gets better when she is two. This is out to be a general phenomenon as shown in Lemma 1.

Lemma 1. *If an agent i can successfully manipulate the DA-IP mechanism at a finite economy E , then i cannot be born in period 0. In addition, if the DA-IP matchings in economy E and at i 's successful manipulation are η and $\hat{\eta}$ respectively, then the following conditions must be satisfied:*

$$(\hat{\eta}_{t+1}(i), \hat{\eta}_{t+1}(i)) \underbrace{\succ}_1 (\eta_{t+1}(i), \eta_{t+1}(i)) \underbrace{\succ}_2 (\eta_t(i), \eta_t(i)) \underbrace{\succ}_3 (\hat{\eta}_t(i), \hat{\eta}_t(i)). \quad (1)$$

Proof. See Appendix C. □

This lemma shows that to manipulate the DA-IP mechanism successfully one will have to accept a weakly worse allocation when she is young in order to improve her future assignment. This is

indeed true even in the Danish priority system in which *IPA* is not satisfied. However, there is a very important difference in terms of the information required for manipulation. But before discussing this let us present an example of a system with the Danish priority structure in which the DA-IP is manipulated.

Example 3 (Manipulability of the DA-IP). *Consider the following example: there are 2 schools $\{s_1, s_2\}$ and each school has a capacity of one agent. Suppose $I_0 = \{i_0\}$, $I_1 = \{i_1\}$, $I_2 = \{i_2\}$ and $I_\tau = \emptyset$ for all $\tau \geq 3$. All agents' top choice is s_1 , but worst choice is h . Each agent's preferences are strongly rankable but satisfies the following condition*

$$(h, s_1) \succ (s_2, s_2).$$

In addition suppose that

$$x_{i_0}^s > x_{i_2}^s > x_{i_1}^s$$

for each $s = s_1, s_2$. Here we assume that the priority system is Danish.

In this economy, agent i_1 's allocation is (s_2, s_2) under the DA-IP mechanism. Now suppose that i_1 reports that her first choice is s_1 , but second choice is h . In this case, agent i_1 obtains (h, s_1) under the DA-IP mechanism. Hence, agent i_1 has an incentive to manipulate.

Examples 1 and 3 suggest that successful manipulations in different priority systems differ in terms of required “sophistication.” In the Danish system by staying home when an agent is young, she jumps ahead of almost everyone in all schools’ priorities. This manipulation is relatively simple as it only involves one action which is staying home when young which ultimately improves the agent’s priority score relative to others’ scores. On the other hand, manipulating the DA-IP mechanism in systems satisfying *IPA* is rather difficult. To see this let us concentrate in the example 1. When agent i_1 misreports her preferences, the agent who was matched to school s in period $t+1$ under truth telling (in our case i_3) will still have priority over i_1 at school s . In other words, agent i_1 's priority score at s does not improve at all no matter what she does. This means that agent i_1 's manipulation must benefit agent i_3 so that she never applies to school s . This of course is possible in the example we considered, but the agent must be rather sophisticated to see through all the possible effects of her manipulation.

5 Simulation Study

In this section, we study how widespread the manipulability of the DA-IP mechanism is using randomly generated data. In particular, we estimate the percentage of markets in which a given agent can manipulate the DA-IP mechanism. To do this, one first need to figure out whether a given agent has a successful manipulation of the DA-IP. A brute force – checking all possible preference reports of an agent – does not take us far because there are $(|S|!)^2$ number of preference reports (the number of isolated preference reports in each period is $|S|!$). Of course, the brute force approach

is not feasible in the static marriage and the many-to-one matching problems, but there are two types of strategies – truncating and dropping – that are sufficient to check whether an agent has a successful manipulation of the static DA mechanism. Unfortunately, as our Example 2 shows, focusing on such strategies could miss some successful manipulations. Thus, we propose a new algorithm to check if an agent has a successful manipulation of the DA-IP in the next subsection.

We believe that the algorithm that we will present can be useful in other contexts. For instance, when the DA is used in static settings one can estimate the impact of an agent on the other agents using our algorithm. Also one perhaps can study different properties of the DA mechanism. For instance, it is well-known that the DA mechanism is bossy²⁷, i.e., an agent sometimes can misreport her preference and alter others’ allocation without changing her own allocation. With the help of our algorithm one can figure out how often an agent has a bossy manipulation.

5.1 Theoretical Results

In this subsection, we present a simplified version of our algorithm to check whether an agent has a successful manipulation of the DA-IP. The interested readers can find the full version of our algorithm, used in our simulation, in Appendix D.

We first note that for any two-year-old agent the DA-IP mechanism is a static DA mechanism. Thus, no two-year old agent has an incentive to misreport her isolated-preference because the DA mechanism is strategy-proof in the static school choice problem. As a result, any manipulating agent must misreport her isolated-preference when she is one year old. Secondly, each manipulating agent’s submitted isolated-preference report at the period she is one must alter the period matching of that period. Otherwise, no agent’s priority score or isolated preference differ from the ones under truth-telling. This suggests that we need to find all the DA-IP matchings that a one-year-old agent’s isolated-preference reports can lead to. As a first step in this direction, first we identify the schools that i can get matched in period t by reporting some isolated preference reports.

For the remainder of this section, we fix an economy E and an agent i who is one year old in period $t \geq 1$. We are investigating whether i can manipulate the DA-IP at economy E . To ease the presentation we do not mention economy E or that agent i is born period t in any of the results that follow. All the agents other than i report their isolated preferences truthfully (as given in economy E). Furthermore, we will only concentrate on period t as this is the period when i potentially misreports her isolated-preference to manipulate the DA-IP mechanism. Thus, unless otherwise stated, we discuss only period- t DA-IP mechanism and period- t isolated-preference reports of the agents with respect to the previous period’s DA-IP matching η_{t-1} . Thus, we simplify the notations by writing P_i instead of $P_i(\eta_{t-1})$.

We say a school s is attainable (for agent i) if there exists an isolated-preference report P'_i such that the DA-IP mechanism allocates i to s in period t if i submits P'_i (while the other school-age agents in period t submit their isolated-preference reports, truthfully). A school is non-attainable if it is not attainable. We reserve the notations S^A and S^{NA} for the set of attainable and non-

²⁷See Kojima (2010).

attainable schools, respectively. The lemma below is the basis of our algorithm to find all attainable and non-attainable schools.

Lemma 2. *Let s be an attainable school. The DA-IP allocates i to s if and only if i submits an isolated-preference report which list only non-attainable schools ahead of s .*

Proof. See Appendix D. □

We now present a simple algorithm to find all the non-attainable and attainable schools.

The Algorithm to Find the Set of Attainable and Non-attainable Schools

Round 1. Agent i updates her true isolated-preference report by placing h at the very end of her list. Find the DA-IP matching for this case and call this matching μ_t^1 . Set $S_1^A = \{\mu_t^1(i)\}$.

Round 2. Agent i updates her report from the previous period by only placing $\mu_t^1(i)$ at the very end of her list. Find the DA-IP matching for this case and call this matching μ_t^2 . Set $S_2^A = S_1^A \cup \{\mu_t^2(i)\}$.

Round k . Agent i updates her report from the previous period by only placing $\mu_t^{k-1}(i)$ at the very end of her list. Find the DA-IP matching for this case and call this matching μ_t^k . Set $S_k^A = S_{k-1}^A \cup \{\mu_t^k(i)\}$.

The algorithm stops at the very first round k^* at which i is allocated to h . The set of attainable schools is $S^A = S_{k^*}^A$ and $S^{NA} = S \setminus S^A$.

We note that h is attainable because child i is assigned to h if she ever applies to h at any round of the DA-IP algorithm. Due to Lemma 2, the algorithm finds the highest ranked attainable school in i 's report in a given round. In addition, the algorithm is constructed so that i 's report in any round ranks the attainable school found in the preceding round after h . As a result, the algorithm not only yields a new attainable school in each round but also finds all the attainable schools in the exact order that were listed in i 's preference report of round 1. Consequently, the number of rounds that the algorithm stops at is the number of attainable schools.

We now look for ways to find all the DA-IP matchings under which i is allocated to some attainable school s . In Lemma 2 we showed that if i is to get allocated to s then she has to rank s as the highest ranked attainable school in her submitted isolated-preference report. In other words, the set of schools i ranks ahead of s must be a subset of the non-attainable schools. It is not complicated to see that the DA-IP mechanism produces the same matching for two reports of i that rank the same set of non-attainable schools ahead of s and in which s is the highest ranked attainable school.²⁸ Thus, the maximal number that one needs to run DA-IP mechanism in order

²⁸One can easily see that each DA-IP matching corresponding to one of these isolated preferences of i matching is

to find all the period- t DA-IP matchings under which i is allocated to some attainable school s is $2^{|S^{NA}|}$ – the number of all subsets of S^{NA} . This number is obviously large if there are many non-attainable schools. Fortunately, it turns out that where i ranks some non-attainable schools in her isolated-preference report does not affect the resulting DA-IP matching as long as i ranks s as the highest attainable school. Thus, for each attainable school s , we now split the non-attainable schools into two groups: redundant and non-redundant.

Definition 7. A school s' is redundant for $s \in S^A$ if the DA-IP mechanism produces the same matching when i submits any two isolated-preference reports, P_i^s and \tilde{P}_i^s ,

1. that rank s as the highest attainable school
2. that the sets of schools ranked higher than s under P_i^s and \tilde{P}_i^s differ only in that one under P_i^s does not contain s' while the one under \tilde{P}_i^s does.

A school s' is non-redundant for s if s' is non-attainable and not redundant for s . We use the notations $S^R(s)$ and $S^{NR}(s)$ to denote the redundant and non-redundant schools for s , respectively.

Although the idea of redundant school is intuitive, checking if a school is redundant using the formal definition is a challenge. However, it turns out that there is an alternative definition which is easy to work with. To present this definition, we need some more notation. Fix a attainable school s , and we write \hat{P}_i^s to denote an isolated-preference report of i in which s is the first ranked school. We write $\hat{\mu}_i^s$ to denote the DA-IP matching when i reports \hat{P}_i^s .

Lemma 3. A school s' is redundant for $s \in S^A$ if and only if the priority score of agent i at school s' is lower than the priority score of those who are matched to s' under $\hat{\mu}_i^s$, i.e.,

$$X_i^{s'}(\eta_{t-1}) < \min_{j \in \hat{\mu}_i^s(s')} \{X_j^{s'}(\eta_{t-1})\}.$$

Proof. See the proof of Lemma 8 in Appendix D. □

Let us pause here to explain the main intuition of the above lemma. First we argue that if a non-attainable school s' does not satisfy the condition described in the lemma above, then i 's two reports, \hat{P}_i^s and \tilde{P}_i^s which ranks s' first and s second, lead to different DA-IP matchings. Otherwise, the DA-IP produces $\hat{\mu}_i^s$ for \tilde{P}_i^s , but in the static economy corresponding to this report of i , matching $\hat{\mu}_i^s$ is not stable (in the static sense) because i ranks s' ahead of s under \tilde{P}_i^s and s' is matched to an agent who has a lower priority than i at $\hat{\mu}_i^s$. This would contradict that the DA mechanism is stable in static school choice problems. Consequently, s' is a non-redundant school. If s' , on the other hand, satisfies the condition identified in the lemma, then s' is non-redundant. Proving this

statically stable in the economy corresponding to the other isolated preference of i . It is well-known that each DA-IP matching in any static economy is agent optimal stable matching in that economy. Combining this with the fact that agent i is matched to school s at both DA-IP matchings, we find that each DA-IP matching Pareto dominates the other. This of course means that the two DA-IP matchings must be the same.

statement is somewhat lengthy, so here we instead show that i 's reports \hat{P}_i^s and \tilde{P}_i^s , lead to the same DA-IP matching, $\hat{\mu}_i^s$. The discussion above implies that $\hat{\mu}_i^s$ is stable (in the static sense) when i 's isolated-preference is \tilde{P}_i^s . At the same time, it is not complicated to see that the DA-IP matching when i 's isolated-preference is \tilde{P}_i^s is also stable in the economy in which i 's isolated-preference is \hat{P}_i^s . Now using the lattice structure of the stable matchings in static settings and the fact that the DA matching is the agent optimal stable matching, one finds that $\hat{\mu}_i^s$ is the DA-IP matching when i 's isolated-preference is \tilde{P}_i^s .

Now we are ready to present our algorithm to find all the period- t DA-IP matchings under which i is allocated to a given attainable school.

The Algorithm to Find the Set of Period- t Matchings under which i is Allocated to a Given attainable School

Fix a attainable school s .

Round 0. Fix a isolated-preference report of i in which s is ranked first. Find the DA-IP matching when i submits this preference report, and let $\mathcal{M}_t^0(s)$ be the set that consists of this matching. Find all the non-redundant schools for s , i.e., $S^{NR}(s)$. Let $\mathcal{S}^{NR}(s) = \{S : S \subseteq S^{NR}(s)\}$.

Round 1. For each $S' \in \mathcal{S}^{NR}(s)$, fix an isolated-preference report of i in which s is the highest ranked attainable school and in which the set of schools that are ranked higher than s is S' . For each of these fixed reports of i , find the DA-IP matchings and denote the set of these matchings by $\mathcal{M}_t(s)$.

Proposition 1. *The algorithm above yields all the period- t matchings that are the result of the DA-IP mechanism for some report of i and in which i is allocated to s .*

Proof. See the proof of Proposition 3 in Appendix D. □

Now we are finally ready to present our algorithm to check the manipulability of the DA-IP mechanism in a given economy by a given agent. Fix an economy E and an agent i . Suppose that agent i is born in period $t \geq 1$.

The Algorithm to Check the Manipulability of the DA-IP Mechanism

Step 1. Run the DA-IP mechanism in economy E until period $t + 1$ and find the DA-IP matching of i .

Step 2. Find the set of attainable schools of i in period t .

Step 3. Consider the attainable schools sequentially. For a fixed attainable school s , find the set of period- t matchings, $\mathcal{M}_t(s)$, under which i is allocated to s . Consider each $\mu_t \in \mathcal{M}_t(s)$

sequentially and find the period $t + 1$ DA-IP matching, μ_{t+1} , assuming that every school-age agent j reports her period $t + 1$ isolated preferences as $P_j(\mu_t)$. If $\mu(i) \succ_i \eta(i)$, then stop the algorithm. In this case, i can manipulate the DA-IP mechanism. Otherwise, consider the next matching in $\mathcal{M}_t(s)$. If the algorithm does not stop before exhausting all the attainable schools and each period- t matchings under which i is matched to an attainable school, then i cannot manipulate the DA-IP mechanism.

5.2 Simulation Results

In this section, we generate markets randomly and then estimate the percentage of the markets in which a given agent has a successful manipulation of the DA-IP mechanism. Given that each agent participates in the assignment system twice, we only consider a two period version of our model. In our simulation exercise, half of the agents are born in period 1 and the other half in period 2. Half of the schools does not admit any agents in period 1 because we here assume that the two year old agents in period 1 will take the seats in these schools. On the other hand, all the schools admit agents in period 2. We vary the number of schools, the schools' capacities and the number of the agents. However, we assume that each market is balanced, i.e., the number of agents that the schools can admit in each period is the same as the number of the school-age agents in that period.

The agents' payoff from holding a spot at a school (including homecare) is drawn according to a uniform distribution on the $[0, 1]$ interval. Each agent's payoff from attending two (not necessarily different) schools in two different periods is the sum of the payoffs the agent obtains by holding a spot at these schools. We are assuming here implicitly that the agents' preferences are strongly rankable. The agents' priority scores are also drawn according to a uniform distribution on the $[0, 1]$ interval. The agents' priorities satisfy IPA.

In our simulation exercise we vary the number of schools and the schools' capacities. For each combination of the number of schools and capacity, we randomly generate 10,000 markets, and in each market we check whether agent #1 can manipulate the DA-IP mechanism using the algorithm we proposed in the previous section. In Table 1 we present the percentage of markets in which agent #1 has a successful manipulation.

Schools' Capacity	10 schools	50 schools	100 schools
1	0.27%	1.05%	1.58%
5	0.12%	0.32%	0.31%
10	0.02%	0.04%	0.28%
15	0.01%	0.05%	0.21%
20	0%	0.03%	0.04%

Table 1: The percentage of markets in which a given agent can manipulate the DA-IP mechanism.

Table 1 shows that the percentage of markets in which a given agent can manipulate DA-IP mechanism is low in general, and this number drops sufficiently close to 0 as the schools' capacities

increase. For instance, if the schools' capacities are 20, then agent #1 can manipulate in out of the 10,000 markets with 100 schools. The capacity of 20 agents is obviously small if we consider the typical schools in the US. This result suggests that in the school choice problem with sibling priorities, the DA-IP mechanism is approximately strategy-proof in practice. The same conclusion holds for the school choice problem in which the students' mobility is accounted. In the day care problem, a typical day care institution has a capacity of more than 20 agents. As a result, in the day care assignment problem, the manipulation of the DA-IP is unlikely.

In the simulations considered in Table 1, any agent's payoff from staying home is drawn according to a uniform distribution on the $[0, 1]$ interval. Consequently, many agents rank home ahead of some schools, which could be somewhat restrictive and affect the manipulation percentage. Thus, we repeat our simulation exercise by assuming that each agent's payoff from staying home is 0. For each combination of the number of schools and capacity, we randomly generate 10,000 markets and report the results in Table 2.

Schools' Capacity	10 schools	50 schools	100 schools
1	1.07%	2.99%	3.21%
5	0.36%	0.99%	0.93%
10	0.33%	0.56%	0.5%
15	0.16%	0.32%	0.28%
20	0.15%	0.27%	0.37%

Table 2: (Homecare as Worst Option) The percentage of markets in which an agent can manipulate the DA-IP Mechanism where the home is the worst option.

The percentage of markets in which an agent can manipulate indeed increased significantly. The main reason behind this result is that when each agent ranks home as the worst option, the number of possible matchings an agent can induce by misreporting her preferences increases significantly. When a child ranks a school ahead of the school she obtains under truth telling, it generally leads to a sequence of rejections and new applications in the DA algorithm. When home is not the worst option, the chances that this sequence ends with some agent choosing home increase greatly. Therefore, the increase of the manipulable markets is expected under the simulations. However, the manipulation percentage still converges to 0 as the capacities of schools increases. In fact, already when the schools' capacities are 20, the manipulation percentage is negligible.

We next consider cases in which the agents have similar preferences. We assume that the schools which are recruiting agents in period 1 are worse than the schools that are not recruiting. The idea here is that because the better schools are highly demanded, their spots are filled with the agents from the previous period. Specifically, the agents' payoffs for the schools which are recruiting in period 1 are drawn according to a uniform distribution on the $[0, 0.5]$ interval while the ones for the schools which are not recruiting in period 1 are drawn according to a uniform on the $[0.5, 1]$ interval. As in the previous case, each agent payoff from home is set to 0. We report our results in Table 3, and for each case we generated 10,000 markets.

Here the percentage of the manipulable markets is in general small. The main reason is that in

Schools' Capacity	10 schools	50 schools	100 schools
1	0.27%	1.04%	1.46%
5	0.12%	0.63%	0.3%
10	0.05%	0.35%	0.31%
15	0.06%	0.15%	0.27%
20	0.04%	0.11%	0.14%

Table 3: (Similar Preferences): The percentage of markets in which an agent can manipulate the DA-IP Mechanism when schools recruiting in period 1 are worse than schools recruiting in period 2.

period 2, the competition for the schools which were closed in period 1 is much fiercer than before because these schools are ranked better than the others for everyone. Because of this, even when an agent succeeds to change period 1 matchings, simply because of high competition in period 2 the chances that he gets a better school is low. Here again the trend that the manipulability percentage decreasing as the schools' capacities increase stays the same.

Finally, we emphasize that in all of our simulations we assumed that the agent who is contemplating to misreport her preference knows everyone else's preferences. This, of course, is a very strong assumption in reality, implying that manipulation might be even less likely than what our simulations show. Indeed, when we study the question of how one's manipulation performs if there is incomplete information, we obtain that manipulation is often a risky strategy, if not unambiguously worse than truth-telling. Specifically, we assume that an agent who is about to misreport her preference knows the payoffs and priorities of the agents who are born at the same time period as her, but not of those who are born in other periods. We first fix such an agent and consider markets with 100 schools that have one spot each. We generated 5,000 markets randomly as we did in our baseline simulation and found that there were 74 markets in which the agent could manipulate the DA-IP mechanism. Then, to introduce incomplete information, for each of these 74 markets we generated 5,000 new variations of these markets in which for each new variation we fix the payoffs and priorities for the agents born in period 1 while randomly generating new payoffs and priorities for the agents who arrive in the second period.²⁹ Thus, we can now calculate the expected payoff of the agent under truth telling and any other strategy. To avoid the computational complexity, we focus on the reports that were successful manipulations in the original 74 markets instead of all the possible preference reports. The expected payoff of the agent from the manipulation exceeded the one from truth telling in 27 out of the 74 markets. However, the difference in the expected payoffs was at most 0.0026 or 0.13%. In general, truth telling dominated the manipulation strategies: if we consider all the possible 370,000 markets (74 x 5,000), the payoff difference between truth telling and manipulation was 0.0103 on average (truth-telling was 0.55% more profitable on average). Furthermore, the agent's manipulation in the original markets did strictly better than truth telling in

²⁹We generate priorities by first drawing a priority score for each agent from a uniform distribution between $[0,1]$. The priority ordering is simply the ordering of these priority scores. The payoffs are drawn according to the same distribution as before (also uniform in $[0,1]$).

only 5.80% of all the markets, while truth telling strictly dominated the manipulation in 54.41% of the markets. This means that any manipulation of the DA-IP in one specific market in general is not likely to succeed in other markets if the payoffs for the agents who arrive later are randomly drawn. Although our assumption on the payoff distribution is somewhat specific, based on this exercise we believe that the manipulation of the DA-IP in dynamic markets is not a significant problem.

All of our simulation results suggest that the manipulability of the DA-IP mechanism becomes negligible as the school's capacities increase. We prove this result theoretically in the next sections.

6 School Assignment with a Continuum of Agents

This section borrows some of the notation from Azevedo and Leshno (2013) and adapts some of their results to our dynamic environment. In this part we assume that the set of agents born in period t , $\bar{I}_t = t \times \mathcal{R} \times [0, 1)^{m+1}$, is a continuum mass of students. The set of children is thus $\bar{I} = \cup_{t=0}^{\infty} \bar{I}_t = \mathbb{N} \times \mathcal{R} \times [0, 1)^{m+1}$. Let $\bar{\nu}$ be a measure on \bar{I} . We assume that the distribution of the children born in each period is identical, i.e., for any $t \neq \tau$, $\bar{J}_t \subseteq \bar{I}_t$ and $\bar{J}_\tau \subseteq \bar{I}_\tau$ such that child $(t, \succ, \mathbf{x}) \in \bar{J}_t$ if and only if child $(\tau, \succ, \mathbf{x}) \in \bar{J}_\tau$, we have that $\bar{\nu}(\bar{J}_t) = \bar{\nu}(\bar{J}_\tau)$. One can relax this assumption without affecting the main results of the paper, but the notation will be considerably more complicated. Let $\bar{\mathbf{r}}$ be the vector of capacities. A continuum economy is $\bar{F} = (\bar{\nu}, \bar{\mathbf{r}})$. For simplicity, we assume that $\nu(I_t) = 1$, for all $t \geq 0$. We will maintain the following assumption throughout the paper.

Assumption 4 (Strict Priorities). *For any school $s \in S$, the measure of the agents who has the same priority at this school is 0, i.e., $\bar{\nu}(\{i : t_i = t \ \& \ x_i^s = e\}) = 0$ for any $t \geq 0$ and $e \in [0, 1)$.*

The assumption above immediately implies the measure of each agent is 0, i.e., $\nu(\{i\}) = 0$.

We are now ready to present the definition of matching which, as in finite economies, is a collection of period matchings. The definition of a period matching is the same one that Azevedo and Leshno (2013) use in static continuum economies.

Definition 8 (Matching). *A period-0 matching $\bar{\mu}_0$ is a function $\bar{\mu}_0 : \bar{I}_0 \cup S \rightarrow \bar{I}_0 \cup S$ such that $|\bar{\mu}_0(i)| = 1$ and $|\bar{\mu}_0(i)| = \{h\}$, for all $i \in I_0$, $\bar{\mu}_0(h) = I_0$ and $\bar{\mu}_0(s) = \emptyset$, for all $s \neq h$.*

A period matching in period $t \geq 1$, $\bar{\mu}_t$, is a function $\bar{\mu}^t : \bar{I}_t \cup \bar{I}_{t-1} \cup S \rightarrow \bar{I}_t \cup \bar{I}_{t-1} \cup S$ such that

1. *For all $i \in \bar{I}_{t-1} \cup \bar{I}_t$, $|\bar{\mu}_t(i)| = 1$ and $\bar{\mu}_t(i) \subset \bar{S}$*
2. *For all $s \in S$, $\bar{\nu}(\bar{I}_{t-1} \cap \bar{\mu}_t(s)) + \bar{\nu}(\bar{I}_t \cap \bar{\mu}_t(s)) \leq \bar{r}^s$ and $\bar{\mu}_t(s) \subset \bar{I}_{t-1} \cup \bar{I}_t$*
3. *For all $i \in \bar{I}_{t-1} \cup \bar{I}_t$, $i \in \bar{\mu}_t(s)$ iff $s = \bar{\mu}_t(i)$.*
4. *Each period- t matching is right continuous, i.e., for any sequence of agents $\{i^k\} = \{(\tau, \succ, x^k)\}$ where $\tau = t-1, t$ converging to $i = (\tau, \succ, x)$, we can find some large K such that $\bar{\mu}_t(i^k) = \bar{\mu}_t(i)$ for all $k > K$.*

A matching $\bar{\mu}$ is a collection of period matchings: $\bar{\mu} = (\bar{\mu}_0, \bar{\mu}_1, \dots, \bar{\mu}_t, \dots)$.

Requirement 4 rules out a multiplicity of stable matchings that differ only by sets of measure zero. As in the finite economy case, we assume that in period 0 everyone stays home. We use the following notations: $\bar{\nu}(\bar{\mu}_t(s)) \equiv \bar{\nu}(\bar{I}_{t-1} \cap \bar{\mu}_t(s)) + \bar{\nu}(\bar{I}_t \cap \bar{\mu}_t(s))$ and $\bar{\mu}(i) \equiv (\bar{\mu}_{t_i}(i), \bar{\mu}_{t_i+1}(i))$.

Now that we have defined matching in continuum economies, we can define the isolated preferences ($\bar{P}_i(\bar{\mu}_t)$), priority score functions ($\bar{X}_i^s(\bar{\mu})$), threshold scores ($\bar{p}^s(\bar{\mu})$), and mechanism ($\bar{\varphi}$) for continuum economies as we did in finite economies. Furthermore, all the assumptions and notions used in finite economies such as IPA, the Danish priority system, stability, manipulability and strategy-proofness are analogously defined for continuum economies.

The DA-IP Mechanism in Continuum Economies

The DA-IP mechanism for continuum economies associates each continuum economy with the matching that is the result of the DA-IP algorithm which we define below.

Fix a continuum economy \bar{F} . We reserve the notation $\bar{\eta}$ for the matching which is the result of the DA-IP algorithm. By the definition of matching, it must be that $\bar{\eta}_0$ matches each agent in \bar{I}_0 to h . Now the period-1 DA-IP matching $\bar{\eta}_1$ is found by running the following algorithm:³⁰

Set the isolated preferences for each school-age agent i in this period to $\bar{P}_i(\bar{\eta}_0)$. In addition, set the priority score vector of each school s in this period to $\bar{X}^s(\bar{\eta}_0)$.

Round 1: Each school-age agent of period 1 applies to her most preferred school according to her isolated preferences ($\bar{P}_i(\bar{\eta}_0)$ in the case of period 1). For each school s , let \bar{p}_1^{s1} be the minimum priority score such that the measure of the applicants to s with priority scores ($\bar{X}_i^s(\bar{\eta}_0)$ that in the case of period 1) that weakly exceed \bar{p}_1^{s1} does not exceed the capacity of school s , \bar{r}^s . School s rejects all the applicants whose priority score is strictly below \bar{p}_1^{s1} and “holds” the others.

In general, at:

Round k : Each agent who was rejected in the previous round applies to her next choice school according to her isolated preferences ($\bar{P}_i(\bar{\eta}_0)$ in the case of period 1). Each school s considers the pool of agents who it had been holding and the current applicants. For each school s , let \bar{p}_1^{sk} be the minimum priority score such that the measure of the agents in the pool of s with priority scores ($\bar{X}_i^s(\bar{\eta}_0)$ that in the case of period 1) that weakly exceed \bar{p}_1^{sk} does not exceed the capacity of school s , \bar{r}^s . School s rejects those in the pool whose priority score is strictly below \bar{p}_1^{sk} and “holds” the others.

The algorithm terminates when no proposal is rejected and each agent is assigned her final tentative assignment. Let $\bar{\mathbf{p}}_1 = (\lim_{k \rightarrow \infty} \bar{p}_1^{sk})_{s \in S}$, which is the period-1 threshold vector associated with the DA-IP.

³⁰The algorithm converges, even though it may require infinite rounds (see Azevedo and Leshno (2013)).

In period 2, the schools' priority scores are updated based on the period-1 DA-IP matching, $\bar{\eta}_1$. In addition, all the school age agents in this period report their isolated preferences based on the period-1 DA-IP matching, $\bar{\eta}_1$. Now using the algorithm described above, we find the period-2 DA-IP matching, $\bar{\eta}_2$. Let \bar{p}_1 be the threshold vector corresponding to period 1 DA-IP matching.

In each period $t \geq 2$ we can run the above algorithm recursively based on the preceding period's DA-IP matching. Let \bar{p}_t be the threshold vector corresponding to period- t DA-IP matching. Also let $\bar{p} = (\bar{p}_t)_{t=0}^\infty$.

The DA-IP mechanism yields a unique matching in each economy.³¹

6.1 Manipulability of DA-IP in Continuum Economies

Now we consider continuum economies and show that the DA-IP mechanism is strategy-proof. The main intuition here is that no student alone can modify, by manipulating her preferences, the priorities in the second period. That is, the priority score vector is immune to single deviations in an environment in which each student has measure zero.³² Then, the main source of manipulations in dynamic matching markets (which consists in misreporting the preferences in order to change the matching in that period, and thus changing the priority of students in the following period) is not present in economies with a continuum of students.

Theorem 3 (Strategy-Proofness of the DA-IP). *The DA-IP mechanism is strategy-proof in continuum economies.*

Proof. Suppose that agent i can successfully manipulate the DA-IP mechanism in some continuum economy \bar{F} . At the successful manipulation let agent i misreport her preferences as $\succ'_i \neq \succ_i$. Let the economy which results from i 's misreporting be \bar{F}' . Let $\bar{\eta}$ and $\bar{\eta}'$ be the DA-IP matchings in \bar{F} and \bar{F}' , respectively. Let the threshold scores corresponding to the $\bar{\eta}$ and $\bar{\eta}'$ be \bar{p} and \bar{p}' , respectively. Since the two economies differ in only agent i 's preferences and given that the measure of each agent is 0, we have that

$$\bar{p} = \bar{p}'.$$

Because i can manipulate the DA-IP mechanism at \bar{F} , in a similar way to Lemma 1, we obtain that

$$(\bar{\eta}'_{t_i+1}(i), \bar{\eta}'_{t_i+1}(i)) \succ_i (\bar{\eta}_{t_i+1}(i), \bar{\eta}_{t_i+1}(i)) \succeq_i (\bar{\eta}_{t_i}(i), \bar{\eta}_{t_i}(i)) \succeq_i (\bar{\eta}'_{t_i}(i), \bar{\eta}'_{t_i}(i)).$$

Set $s \equiv \bar{\eta}'_{t_i+1}(i)$. The relation above means that i is not matched to s in period t_i at both matchings $\bar{\eta}$ and $\bar{\eta}'$. Consequently, it must be that $\bar{X}_i^s(\bar{\eta}_{t_i}) = \bar{X}_i^s(\bar{\eta}'_{t_i}) = x_i^s$. However, i is matched to s in period $t_i + 1$ at $\bar{\eta}'$ but not at $\bar{\eta}$. Then by the definition of the DA-IP algorithm, it must be that

³¹This result is a straightforward adaptation of the proof by Azevedo and Leshno (2013) (Theorem 1).

³²Azevedo and Leshno (2013) note that small changes in the measure η (or in the capacity of each school) can affect the threshold priority score vector significantly. However, note that a zero-measure agent is not able to affect the threshold scores through unilateral deviations.

$\bar{X}_i^s(\bar{\eta}'_{t_i}) = x_i^s \geq p_{t_i+1}^s$ but $\bar{X}_i^s(\bar{\eta}_{t_i}) = x_i^s < p_{t_i+1}^s$. However, these two inequalities contradict that $p_{t_i+1}^s = p_{t_i+1}^s$. \square

In the theorem above, we assumed that the priorities of the schools satisfy the *IPA* condition. When we consider the Danish priority scoring system, the theorem above no longer holds. The result that an agent –with measure zero– could not affect her priority in the second period, no longer holds here. An agent’s single preference report can place her at the top of the priority ranking of a school in the subsequent period (by reporting homecare today). Informally, under the Danish scoring system, a student can produce discrete changes in the priorities of the following period; if we think of priorities as a function of students’ submitted preferences in the previous period, then, roughly speaking, these results are related to the “continuity” (or lack thereof) of this function.

Theorem 4 (Manipulation under Danish Scoring System). *If the priority scoring system is Danish, then the DA-IP mechanism is manipulable even in some continuum economies.*

Proof. Consider an economy in which the threshold score at some school s corresponding to the DA-IP matching is \bar{p} with $\bar{p}_t^s > 0$, $\bar{p}_{t+1}^s > 0$ and $t \geq 1$. Consider an agent i who was born in period t , and whose preferences satisfy the following two conditions: (i) (s, s) is the most preferred bundle, and (ii) $(h, s) \succ_i (s', s'')$ for all $s' \neq s$ and $s'' \neq s$. In addition, suppose that $x_i < \min\{\bar{p}_t^s, \bar{p}_{t+1}^s\}$. Clearly, agent i does not attend s by reporting her preferences truthfully. However, if she reports s as her first choice and h as her second choice, then she will stay home when she is one but attends s when she is 2. This means that agent i has a profitable manipulation. \square

Here we note that there are some economies in which a positive mass of agents can manipulate the DA-IP mechanism at this economy.

Finally, we present an example that aims to capture the main features of the current Boston pre-school system.³³ In Boston, some of the most demanded kindergarten seats are obtained by agents who enrolled in prekindergarten in those schools. Thus, an agent might benefit from attending prekindergarten even if she would otherwise be better off with another pre-school. Below is a minimalist example that aims to capture these features, showing that the Boston prekindergarten system is manipulable even in a continuum economy. In the example, priorities satisfy IPA.

Example 4 (Boston Pre-School System). *Consider a continuum economy F . In this economy, agents go to preschool when they are 1 and to kindergarten when they are 2. There are three schools, $\{s_1, s_2, s_3\}$, and each school has a finite capacity \bar{r} . Here schools s_1 and s_2 are a preschool and a school, respectively. Thus, s_1 offers only preschool classes only while s_2 offers kindergarten classes. On the other hand, s_3 is an integrated institution that offers both preschool and kindergarten classes. The set of agents born in each period t is \bar{I} and its measure is $2\bar{r}$. The priority structure satisfies IPA. Suppose that agent born in period t , whose preferences are given by: $(s_1, s_3) \succ (s_3, s_3) \succ (s_1, s_2)$. In this economy, the DA-IP mechanism matches the set of agents with measure \bar{r} who have the highest priority at s_1 to s_1 at period t . The others attend s_3 . However, only those who*

³³For a recent story covering the Boston pre-school system, please see Ebbert (2011).

attended s_3 in period t attend s_3 in period $t+1$, due to IPA. Now consider the agents who attended s_1 in period t even though they have higher priority at s_3 than some of those who attend s_3 at period t . Clearly, these agents have an incentive to misreport their preferences so that (s_3, s_3) as their top choice. Thus, the DA-IP is manipulable even in a continuum economy.

7 Large Markets and Convergence

Consider a finite economy $E = (I, \mathbf{r})$. We now define the measure for each finite economy E based on its empirical distribution. Specifically, the measure of each agent i is $\tilde{\nu}(\{i\}) = 1/|I_0|$. On the other hand, let the capacities of the schools be $\tilde{\mathbf{r}} = \mathbf{r}/|I_0|$. Using this empirical distribution, we will denote finite economies in a similar fashion to continuum economies. Specifically, let $\tilde{F} = (\tilde{\nu}, \tilde{\mathbf{r}})$ denote the finite economy E . With this notation we can define the convergence of finite economies to a continuum economy.

Definition 9. *A sequence of finite economies E^k converges to a continuum economy \bar{F} if the sequence of economies $\tilde{F}^k = (\tilde{\nu}^k, \tilde{\mathbf{r}}^k)$ corresponding to E^k satisfies the following two conditions:*

1. $\tilde{\nu}^k$ converges to $\bar{\nu}$ in weak* topology
2. $\tilde{\mathbf{r}}^k$ converges to $\bar{\mathbf{r}}$ in supremum norm.

Here observe that if E^k converges to \bar{F} , then the ratio of the size of agents born in any period t to the size of the agents born in $t-1$ converges to 1.

In this section, we assume the following assumption in order to ensure the convergence of the stable matchings when the finite economies converge to a continuum economy.

Assumption 5 (Market Thickness). *Consider any continuum economy \bar{F} . Then for any $t \geq 0$, any isolated preferences $\bar{P} \in \bar{\mathcal{P}}$ and any $x \ll x' \ll \mathbf{1}$,*

$$\bar{\nu}(\{i : t_i = t \ \& \ \bar{P}_i = \bar{P} \ \& \ x \leq x_i \leq x'\}) > 0.$$

The assumption above means that the market is thick in the sense that the type space is sufficiently rich. This assumption guarantees the uniqueness of stable matchings in continuum economies.

We now consider what happens to the DA-IP matchings when the sequence of economies converges to a continuum economy. To study this we first define the distance between two DA-IP matchings as defined in Azevedo and Leshno (2013). Let η and $\bar{\eta}$ be the DA-IP matchings of a continuum economy F and of a finite economy E . The period t distance between $\bar{\eta}$ and η are as follows:

$$d_t(\eta, \bar{\eta}) = \| p_t - \bar{p}_t \|_\infty.$$

Let $d(\eta, \bar{\eta}) = (d_t(\eta, \bar{\eta}))_{t=0}^\infty$.

Now that the distance between two DA-IP matchings defined, we can consider the convergence of the DA-IP matchings when the sequence of finite economies converges to a continuum economy.

Definition 10. *A sequence of DA-IP matchings, $\{\eta^k\}$, in economies $\{E^k\}$ converges to $\bar{\eta}$ if*

$$\lim_{k \rightarrow \infty} d_t(\eta^k, \bar{\eta}) = 0 \text{ for all } t \geq 1.$$

Now we are ready to present the convergence result of the DA-IP matchings as the sequence of finite economies converges to a continuum economy.

Proposition 2. *If a sequence of finite economies E^k converges to a continuum economy \bar{F} , then the sequence of DA-IP matchings $\{\eta^k\}$ converges to $\bar{\eta}$.*

Proof. See Appendix C. □

We are now finally ready to study how the incentives to manipulate the DA-IP mechanism change as the market size grows. For a finite economy E , let us define L_t as the set of the agents born in period t who benefits by manipulating the DA-IP mechanism in this economy. In the following theorem, we show that as finite economies converge to a continuum economy, the fraction of the agents who can manipulate in finite economies converges to 0.

Theorem 5. *If a sequence of finite economies, $\{E^k\}$, converges to a continuum economy \bar{F} , then $\left\{ \frac{|L_t^k|}{|I_t^k|} \right\} \rightarrow 0$ for each $t \geq 1$.*

Proof. See Appendix C. □

The main idea of the theorem above is the following: in Lemma 1 we showed that to successfully manipulate the DA-IP mechanism an agent must weakly get worse at her first period and get better in her second period. In other words, by manipulating this agent must be able to lower the threshold score corresponding to the DA-IP mechanism in her second period. However, as the economy grows this agent becomes a tiny part of the economy, and at some point her manipulation will have minuscule impact on the threshold scores because these are converging to a fixed value, due to Proposition 2. Thus, if an agent can manipulate the DA-IP mechanism in a large economy then her priority score at the school she manages to attain as a result of the manipulation must be very close the threshold score of this school.

Remark 1. *Recall that we assumed that in period 0 every agent stays home. This assumption can be relaxed, but one extra minor assumption must be made not to affect our convergence results. As we already mentioned in Footnote 2.1 we take μ_0 and $\bar{\mu}_0$ as the primitives of our model. To preserve our main results, we need to assume that the sequence of period-0 matchings in finite economies, $\{\mu_0^k\}$, converges to $\bar{\mu}_0$ as E^k converges to \bar{F} .*

Remark 2. *The market thickness assumption (Assumption 5) can be weakened significantly in Theorem 5. Specifically, once the DA-IP matching is fixed we can define a static continuum economy*

in each period in which the school age children in that period participate and their preferences are determined by their isolated preferences.³⁴ If each such economy has a unique stable (in the static sense) then the main result of this paper is still valid. Azevedo and Leshno (2013) show that in a wide range of static economies has a unique stable matching. In this sense, the approximate strategy-proofness of the DA-IP matching is valid in a vast array of economies.

Remark 3. Theorem 5 is not valid if one disposes of the market thickness assumption and if the economy violates the condition discussed in the previous remark. We show this point in Appendix C by presenting an example in which the fraction of agents who can manipulate the DA-IP mechanism does not converge to 0 as a sequence of finite economies converge to such a continuum economy.

As a last remark, we note that a version of theorem 4, which states that the DA-IP mechanism is manipulable even in continuum economies when priorities are Danish, also holds for large, finite markets. In other words, if the priority scoring system is Danish, then the DA-IP is manipulable even as the market becomes large.

8 Conclusion

In this paper we have studied the strategic incentives in the DA mechanism in dynamic matching markets, analyzing how manipulable the DA mechanism is in a dynamic school choice model. We first proved that if each school's priority is affected by the previous period's matching only through previously enrolled agents, then the period-by-period DA mechanism is approximately strategy-proof when the schools' capacities as well as the number of participating agents is large. Our simulation results show that the manipulability issue of the DA mechanism is practically negligible when the schools' capacities are around twenty. This number is small compared to the actual capacities of the schools in practice. Thus, our paper provides another justification for the use of the DA mechanism in the school choice problem and also provides support for its use in other dynamic matching problems, such as the assignment of young children to public day care centers and the assignment of teachers to public schools.

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³⁴See Appendix C for more information.

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Appendix A: Static Stability in Small Economies

To prove Proposition 2 we need some new definitions and results which we include in Appendices A and B. The proof of Proposition 2 is in Appendix C.

Fix a finite economy $E = (I, r)$ and a period $t - 1$ matching μ_{t-1} of this economy. Now let us construct a new period- t finite economy $E_t(\mu_{t-1})$ based on our original economy and μ_{t-1} . In this new economy the set of agents is $I_t \cup I_{t-1}$ and each agent i is defined by a pair $(P_i(\mu_{t-1}), X_i(\mu_{t-1}))$.

Observe that no two agents i and j in this static period- t economy $E_t(\mu_{t-1})$ can have the same priority score which is less than 1, i.e., it cannot be that $X_i^s(\mu_{t-1}) = X_j^s(\mu_{t-1}) < 1$ for any s .

Definition 11 (Static Stability). *We say that school-agent pair (s, i) blocks a period t matching μ_t in economy $E_t(\mu_{t-1})$ if there exists a school-agent pair (s, i) such that*

1. $sP_i(\mu_{t-1})\mu_t(i)$,
2. $|\mu_t(s)| < r^s$ or/and $X_i^s(\mu_{t-1}) > X_j^s(\mu_{t-1})$ for some $j \in \mu_t(s)$.

A matching μ_t is statically stable in economy $E(\mu_{t-1})$ if no school-agent pair blocks μ_t .

From Gale and Shapley (1962) each period $t \geq 0$ DA-IP matching η_t is statically stable in economy $E_t(\eta_{t-1})$.

We state the following lemma which is needed later in Appendix D.

Lemma 4. *Consider two static economies in period t , E_t and E'_t , which are identical except that the set of schools preferred to some school s for some agent i in economy E_t is a subset of the one in economy E'_t . Let μ'_t be a statically stable matching in E'_t . If $\mu'_t(i) = s$, then μ'_t is a statically stable matching in E_t .*

Proof. Recall that every agent has the same priority score in the two economies. In addition, each agent $j \neq i$ has the the isolated preferences in the two economies. These and the fact that μ'_t is stable in economy E' imply that no school s' and agent $j \neq i$ can block μ'_t in economy E_t . Suppose that i and some school s' blocks μ'_t in economy E_t . Then i must have a higher priority at s' in E_t . Then, by the conditions given in the lemma, i must have a higher priority at s' in E'_t . Therefore, i and s' should have been able to block μ'_t in E'_t , which contradicts that μ'_t is stable in E'_t . Thus, i cannot be a part of a blocking pair. \square

Appendix B: Static Stability in Continuum Economies

Let us fix an economy $(\bar{\nu}, \bar{r})$ satisfying Assumption 5. Fix any period $t \geq 0$ and a period $t - 1$ matching $\bar{\mu}_{t-1}$. Now let us construct a new period t continuum economy $\bar{F}_t(\mu_{t-1}) = (\bar{\nu}_t, \bar{r})$ based on our original economy and $\bar{\mu}_{t-1}$. In this new economy the set of agents is $\bar{I}_t \cup \bar{I}_{t-1}$ and each agent i is defined by a pair $(\bar{P}_i(\bar{\mu}_{t-1}), \bar{X}_i(\bar{\mu}_{t-1}))$. With the new notations, $\bar{I}_t \cup \bar{I}_{t-1}$ is distributed on $\bar{\mathcal{P}} \times [0, 1]^n$ according to a measure $\bar{\nu}_t$ where

$$\begin{aligned} \bar{\nu}_t(\{i \in \bar{I}_{t-1} \cup \bar{I}_t : x \leq \bar{X}_i(\bar{\mu}_{t-1}) \leq x'\}) &= \bar{\nu}(\{i \in I_{t-1} : x \leq \bar{X}_i(\bar{\mu}_{t-1}) \leq x'\}) \\ &+ \bar{\nu}(\{i \in I_t : x \leq \bar{X}_i(\bar{\mu}_{t-1}) \leq x'\}) \end{aligned}$$

for all $x, x' \in [0, 1]^n$ where $x \ll x'$.

Let $\bar{\mathcal{P}}$ be the all possible rankings of S . Now observe that $\bar{\nu}_t$ has a full support because by Assumption 5 it must be that

$$\bar{\nu}(\{i \in I_t : \bar{P} = \bar{P}(\mu_{t-1}) \& x \leq \bar{X}_i(\bar{\mu}_{t-1}) \leq x'\}) > 0$$

for all $\bar{P} \in \mathcal{P}$ and $x, x' \in [0, 1]^n$ where $x \ll x'$. In addition, $\bar{\nu}_t(\{i \in \hat{I}_t : X_i^s(\bar{\mu}_{t-1}) = x\}) = 0$ for all $x < 1$ and $s \in S$.

Definition 12. *Period t matching $\bar{\mu}_t$ is statically stable in economy $\bar{F}_t(\bar{\mu}_{t-1})$ if there exists no school-agent pair (s, i) such that*

1. $s\bar{P}_i(\bar{\mu}_{t-1})\bar{\mu}_t(i)$,
2. $\hat{\nu}(\mu_t(s)) < \bar{r}^s$ or/and $\bar{X}_i^s(\bar{\mu}_{t-1}) > \bar{X}_j^s(\bar{\mu}_{t-1})$ for some $j \in \bar{\mu}_t(s)$

Lemma 5. *For any economy $\bar{F}_t(\bar{\mu}_{t-1})$, there exists a unique statically stable matching.*

Proof. We already pointed out that $\bar{\nu}_t$ has a full support and $\bar{\nu}_t(\{i \in \bar{I}_{t-1} \cup \bar{I}_t : \bar{X}_i^s(\bar{\mu}_{t-1}) = x\}) = 0$ for all $x < 1$ and $s \in S$. Therefore, all the requirements for Theorem 1 of Azevedo and Leshno (2013) is satisfied, hence $\bar{F}_t(\bar{\mu}_{t-1})$ has a unique statically stable matching. \square

Lemma 6. *For any economy $\bar{F}_t(\bar{\eta}_{t-1})$, $\bar{\eta}_t$ is a unique statically stable matching.*

Proof. This is a direct consequence of Lemma 6 and Proposition A1 of Azevedo and Leshno (2013). \square

Appendix C: Proofs

Proof of Lemma 1. First let us show that any agent born in period 0 cannot manipulate the DA-IP mechanism. To see this, recall that these agents' matching in period 0 is exogenously determined and to determine the period-1 matchings, the DA-IP mechanism uses the isolated preferences. In addition, because the DA mechanism is strategy proof in static settings, by misreporting no agent born in period 0 improves in terms of her isolated preferences.

Relation 3 in (1) follows directly from the fact that the DA-IP mechanism is strategy-proof in terms of isolated preferences. Agent i has the highest priority at school $\eta_{t_i}(i)$ in period $t_i + 1$. Thus, the definition of isolated preferences and the definition of the DA-IP yield that

$$\eta(i) \succeq_i (\eta_{t_i}(i), \eta_{t_i}(i)). \quad (2)$$

(2) and Assumption 1 yield Relation 2 in (1).

Now we show relation 1. On contrary suppose $(\eta_{t_i+1}(i), \eta_{t_i+1}(i)) \succeq_i (\hat{\eta}_{t_i+1}(i), \hat{\eta}_{t_i+1}(i))$. First observe that $\hat{\eta}_{t_i+1}(i) \neq \hat{\eta}_{t_i}(i)$. Otherwise, relation 3 and (2) yield that $\eta(i) \succeq_i \hat{\eta}(i)$ which is a contradiction. Because $\hat{\eta}_{t_i+1}(i) \neq \hat{\eta}_{t_i}(i)$, Assumption 1 and relation 3 yield that

$$(\eta_{t_i}(i), \hat{\eta}_{t_i+1}(i)) \succeq_i \hat{\eta}(i). \quad (3)$$

Now observe that $\hat{\eta}_{t_i+1}(i) \neq \eta_{t_i}(i)$. Otherwise, (3) gives that $(\eta_{t_i}(i), \eta_{t_i}(i)) \succeq_i \hat{\eta}(i)$. This and (2) give that $\eta(i) \succeq_i \hat{\eta}(i)$, which is a contradiction. Because $\eta_{t_i}(i) \neq \hat{\eta}_{t_i+1}(i)$, the supposition and Assumption 1 give

$$\eta(i) \succeq_i (\eta_{t_i}(i), \hat{\eta}_{t_i+1}(i)).$$

This and (3) yield that $\hat{\eta}_{t_i+1}(i) = \eta_{t_i+1}(i)$. Then the fact that $\hat{\eta}_{t_i+1}(i) \neq \hat{\eta}_{t_i}(i)$, Relation 3 and Assumption 1 yield that $\eta(i) \succeq_i \hat{\eta}(i)$ which is a contradiction. \square

Proof of Proposition 2. Take any sequence of DA matchings $\{\eta^k\}$ and the corresponding sequence of threshold scores $\{\mathbf{p}^k\}$. For this proof we will use an induction argument. Assume that for all $\tau = 0, \dots, t-1$, $\mathbf{p}_\tau^k \rightarrow_{k \rightarrow \infty} \bar{\mathbf{p}}_\tau$. At $t = 1$, this is definitely true because $\mathbf{p}_0^k = 0$ and $\bar{\mathbf{p}}_0 = 0$. Now we show $\mathbf{p}_t^k \rightarrow_{k \rightarrow \infty} \bar{\mathbf{p}}_t$.

As in Appendices A and B, we construct period- t economies $\{E_t^k(\eta_{t-1}^k)\}$ and $\bar{E}_t(\bar{\eta}_{t-1})$. Now based on $E_t^k(\mu_{t-1}^k)$ let us define economy $\tilde{E}_t^k(\mu_{t-1}^k) = (\tilde{\nu}, \tilde{r})$ where the measure $\tilde{\nu}$ is a measure satisfying $\tilde{\nu}(\{i\}) = 1/|I_0^k|$, and $\tilde{r} = r/|I_0^k|$. Because $\eta_{t-1}^k \rightarrow_{k \rightarrow \infty} \bar{\eta}_{t-1}$, any sequence $\{\mathbf{p}_{t-1}^k\}$ converges to $\bar{\mathbf{p}}_{t-1}$. Consequently, the sequence of measures $\tilde{\nu}_t^k$ must converge to $\bar{\nu}_t$ in the weak* sense. Then Theorems 2(ii) and 2(iii) of Azevedo and Leshno (2013) give that $\mathbf{p}_t^k \rightarrow \bar{\mathbf{p}}_t$. This completes the proof. \square

Proof of Theorem 5. Suppose that agent i in finite economy E can manipulate the DA-IP mechanism. Let agent i 's DA-IP matchings in economy E and at the successful manipulation be $\eta(i)$ and $\hat{\eta}(i)$, respectively. Let \mathbf{p} and $\hat{\mathbf{p}}$ be the threshold scores corresponding to η and $\hat{\eta}$, respectively.

Lemma 1 implies that agent i is not matched to $s \equiv \hat{\eta}_{t_i+1}(i)$ in period t_i at both matchings $\hat{\eta}$ and η . Thus, $X_i^s(\eta_{t-1}) = X_i^s(\hat{\eta}_{t-1}) = x_i^s$. Then because i is matched to s at period $t_i + 1$ under $\hat{\eta}$ but not under η , it must be that

$$\hat{p}_{t_i+1}^s \leq x_i^s < p_{t_i+1}^s.$$

In other words, if an agent i can manipulate the DA-IP mechanism then there must exist a school $s \in S$ such that the inequality above is satisfied. Therefore, to prove the theorem it suffices to show that at each $t \geq 1$, s and $\epsilon > 0$, there exists large enough \bar{k} such that for all $k \geq \bar{k}$, there exists no agent with $t_i = t - 1$, $|x_i^s - \bar{p}_t^s| \geq \epsilon$ and $\hat{p}_t^{sk} \leq x_i^s < p_t^{sk}$.

Suppose that the statement above is false. This means that for some $t \geq 1$, s , $\epsilon > 0$ and any \bar{k} , there exists $k \geq \bar{k}$ and i with $t_i = t - 1$, $|x_i^s - \bar{p}_t^s| \geq \epsilon$ and $\hat{p}_t^{sk} \leq x_i^s < p_t^{sk}$. In other words, we can choose a subsequence of economies E^{k_j} , such that in each economy in this sequence, there exists agent i^{k_j} who is born in period $t - 1$, $|x_{i^{k_j}}^s - \bar{p}_t^s| \geq \epsilon$ and $\hat{p}_t^{sk_j} \leq x_{i^{k_j}}^s < p_t^{sk_j}$. Clearly, $\{E^{k_j}\}$ converges to \bar{F} in weak* sense. This means that $\hat{p}_t^{sk_j}$ must converge to \bar{p}_t^s . Now consider the sequence of finite economies \hat{E}^{k_j} which differs from E^{k_j} only in that the preferences of agent i^{k_j} is the same as the preferences reported at the successful manipulation. Because in each of these economies only one agent's preferences are changed, \hat{E}^{k_j} converges to \bar{F} in weak* sense. This means that $\{\hat{p}_t^{sk_j}\}$ must converge to \bar{p}_t^s . Recall that we already showed that $p_t^{sk_j}$ converges to \bar{p}_t^s . This means that as k_j increases, $x_{i^{k_j}}^s$ must be arbitrarily close to \bar{p}_t^s because $\hat{p}_t^{sk_j} \leq x_{i^{k_j}}^s < p_t^{sk_j}$. Therefore, for a high enough k_j it cannot be $|x_{i^{k_j}}^s - \bar{p}_t^s| \geq \epsilon$ which is a contradiction.

This completes the proof as $\bar{\nu}(\{i : t_i = t \& x_i^s = e\}) = 0$ for any t and $e \in [0, 1)$. \square

One can relax Assumption 5 in Theorem 5. For instance, Assumption 5 can be replaced in Theorem 5 by the following condition: for all $t \geq 1$, $\bar{\eta}_t$ is a unique statically stable matching in a static economy $\bar{F}_t(\bar{\eta}_{t-1})$. Observe here that this new condition is much weaker than requiring that economy \bar{F} has a unique stable matching.

Finally, we now show that if $\bar{\eta}_t$ is not a unique statically stable matching in a static economy $\bar{F}_t(\bar{\eta}_{t-1})$ for some $t \geq 1$ then the fraction of agents who can manipulate the DA-IP mechanism does not necessarily converge to 0. In this example, we relax some of the assumptions we used in the paper to simplify the presentation. Specifically, we will not assume that the agents born in each period are distributed identically in the continuum economy $\bar{F} = (\bar{\nu}, \bar{r})$ to which the sequence of finite economies converges. In addition, the measure of the agents born in each period is not necessarily 1. Finally, of course in one of the periods the DA-IP matching of that period will not be a unique statically stable matching.

Example 5. *There are 4 schools, s_1, s_2, s_3 and s_4 . Consider the following continuum economy $\bar{F} = (\bar{\nu}, \bar{r})$ such that $\bar{r}^{s_1} = \bar{r}^{s_2} = \bar{r}^{s_3} = \bar{r}^{s_4} = 0.25$, $\bar{\nu}(\bar{I}_0) = 0$, $\bar{\nu}(\bar{I}_1) = 0.5$, $\bar{\nu}(\bar{I}_2) = 0.75$, $\bar{\nu}(\bar{I}_3) = 0.25$ and $\bar{\nu}(\bar{I}_t) = 0$, for all $t > 3$. We assume that each agent's preference satisfies strong rankability. In addition, \bar{I}_1 is partitioned into \bar{I}_1^1 and \bar{I}_1^2 where $\bar{\nu}(\bar{I}_1^1) = \bar{\nu}(\bar{I}_1^2) = 0.25$. Furthermore, \bar{I}_2 is partitioned into \bar{I}_2^1, \bar{I}_2^2 and \bar{I}_2^3 where $\bar{\nu}(\bar{I}_2^1) = \bar{\nu}(\bar{I}_2^2) = \bar{\nu}(\bar{I}_2^3) = 0.25$. The preference rankings of the*

agents satisfies the following conditions:

$$\begin{aligned}
i \in I_1^1 & : (s_1, s_1) \\
i \in I_1^2 & : (s_2, s_2) \\
i \in I_2^1 & : (s_1, s_1) \succ_i (h, h) \succ_i (s_4, s_4) \\
i \in I_2^2 & : (s_4, s_4) \succ_i (s_2, s_2) \succ_i (s_3, s_3) \\
i \in I_2^3 & : (s_3, s_3) \succ_i (s_2, s_2) \succ_i (s_4, s_4) \\
i \in I_3 & : (s_4, s_4) \succ_i (s_1, s_1).
\end{aligned}$$

The priority scores are distributed continuously as follows:

$$\begin{aligned}
i \in I_2^1 & : x_i^{s_4} \in [0.75, 0.85) \ \& \ x_i^{s_1} \in [0.5, 0.75) \\
i \in I_2^2 & : x_i^{s_3} \in [0.75, 1) \ \& \ x_i^{s_4} \in [0.5, 0.75) \\
i \in I_2^3 & : x_i^{s_4} \in [0.85, 1) \ \& \ x_i^{s_3} \in [0.5, 0.75) \\
i \in I_3 & : x_i^{s_4} \in [0.85, 1) \ \& \ x_i^{s_1} \in [0.75, 1)
\end{aligned}$$

For this economy, the DA-IP matching, $\bar{\eta}$, is given in the following table.

	Period 1	Period 2	Period 3
s_1	\bar{I}_1^1	\bar{I}_1^1	\bar{I}_3
s_2	\bar{I}_1^2	\bar{I}_1^2	\emptyset
s_3	\emptyset	\bar{I}_2^3	\bar{I}_2^3
s_4	\emptyset	\bar{I}_2^2	\bar{I}_2^2
h	\emptyset	\bar{I}_2^1	\bar{I}_2^1

Consider now the following matching $\bar{\mu}$:

	Period 1	Period 2	Period 3
s_1	\bar{I}_1^1	\bar{I}_1^1	\bar{I}_2^1
s_2	\bar{I}_1^2	\bar{I}_1^2	\bar{I}_2^2
s_3	\emptyset	\bar{I}_2^2	\bar{I}_2^3
s_4	\emptyset	\bar{I}_2^3	\bar{I}_3
h	\emptyset	\bar{I}_2^1	\emptyset

Clearly, each agent in \bar{I}_2^1 prefers $\bar{\mu}$ to the DA-IP matching $\bar{\eta}$. We will show below that there exists a sequence of finite economies that converges to the continuum economy \bar{F} and such that there is a suitable profitable manipulation for any agent in \bar{I}_1^1 . Note that market thickness is violated in the example and also that μ_t is statically stable matching in economy $\bar{F}(\mu_{t-1})$ for all $t = 1, 2, 3$ and, thus, there are multiple statically stable matching in economy $\bar{F}(\mu_1) = \bar{F}(\eta_1)$.

Now we construct a sequence of economies $\{E^k\}$ that converges to \bar{F} . For each given $k = 1, \dots, \infty$, economy E^k is as follows: $r^{ks_1} = r^{ks_2} = r^{ks_3} = r^{ks_4} = k$. In addition, $|I_0^k| = 0$, $|I_1^k| = 2k$, $|I_2^k| = 3k$, $|I_3^k| = k$ and $|I_t^k| = 0$, for all $t > 3$. We assume that each agent's preference

satisfies strong rankability. In addition, I_1^k is partitioned into I_1^{k1} and I_1^{k2} where $|I_1^{k1}| = |I_1^{k2}| = k$. Furthermore, I_2^k is partitioned into \bar{I}_2^{k1} , I_2^{k2} and I_2^{k3} where $|I_2^{k1}| = |I_2^{k2}| = |I_2^{k3}| = k$. The preference rankings of the agents mirror those in the continuum economy \bar{F} specified above. We set $\tilde{r}^{ks_1} = \tilde{r}^{ks_2} = \tilde{r}^{ks_3} = \tilde{r}^{ks_4} = 0.25 = k/4k$ and $\tilde{v}(\{i\}) = 1/4k$. Furthermore, we set the priority scores so that (1) no two agents have the same priority score (2) the range of the agents' priority scores are the same as in continuum economy \bar{F} , and (3) \tilde{v}^k converges to \bar{v} in weak* topology. With these assumptions, E^k converges to \bar{F} .

Observe here that the DA-IP matching in economy E^k mirrors the one in the continuum economy \bar{F} . However, suppose that an agent in $i^* \in I_2^{k1}$ reports a strongly rankable preference that satisfies

$$(s_1, s_1) \succ_{i^*} (s_4, s_4) \succ_{i^*} (h, h).$$

In this case, the DA-IP matching is as follows: As a result of this manipulation, i^* definitely

	Period 1	Period 2	Period 3
s_1	I_1^{k1}	I_1^{k1}	I_2^{k1}
s_2	I_1^{k2}	I_1^{k2}	I_2^{k2}
s_3	\emptyset	I_2^2	I_2^3
s_4	\emptyset	I_2^3	I_3
h	\emptyset	I_2^1	\emptyset

improves. Therefore, no matter how high k is, all the agents in I_2^{k1} can manipulate DA-IP mechanism. Clearly, the percentage of the children who have a manipulation of the DA-IP mechanism stays constant for each k .

Example 5 has a very specific structure in the sense that in period 2 only one agent's false preference report leads to a matching that is significantly different from the DA-IP matching under truth telling. In other words, if the market is very big this false report leads to a very long chain of rejections and proposals in the DA algorithm. Indeed, in any randomly generated economy, we suspect that one agent's report having a such a big impact is very unlikely. In this sense Example 5 is useful as a theoretical exercise.

Appendix D:

In this Appendix we present the full version of our algorithm that checks whether an agent can manipulate the DA-IP mechanism. We also extend some of the theoretical results discussed in Section 5. Recall that we are working with a fixed economy E and an agent i who is one year old in period $t \geq 1$. We are investigating whether i can manipulate the DA-IP at economy E . To simplify the presentation we do not mention economy E or that agent i is born period t in any of the results that follow. All the agents other than i report their isolated preferences truthfully (as given in economy E).

Lemma 7. *Suppose that agent i has a successful manipulation of the DA-IP mechanism. Let η and $\hat{\eta}$ be the DA-IP matchings at truth telling and at the successful manipulation of i , respectively. In addition, suppose that $\hat{P}_i(\hat{\eta}_{t-1})$ ($\equiv \hat{P}_i(\eta_{t-1})$) and $\hat{P}_i(\hat{\eta}_t)$ are the respective reported isolated preferences of i in periods t and $t + 1$ at this manipulation.*

- (i) *If $\hat{P}(\hat{\eta}_t) \neq P(\hat{\eta}_t)$, then reporting $\hat{P}(\hat{\eta}_{t-1})$ in period t and $P(\hat{\eta}_t)$ in period $t + 1$ is also a successful manipulation for agent i .*
- (ii) *It must be that $\eta_t \neq \hat{\eta}_t$, and $P(\eta_t) \neq P(\hat{\eta}_t)$.*

Proof. (i) Let $\tilde{\eta}$ be the DA-IP matching if i reports $\hat{P}_i(\hat{\eta}_{t-1})$ in period t and $P(\hat{\eta}_t)$ in period $t + 1$ while all the other agents report their isolated preferences truthfully. Now using the strategy proofness of the DA mechanism in static settings, we obtain that

$$\tilde{\eta}_{t+1}(i) R_i(\hat{\eta}_t) \hat{\eta}_{t+1}(i).$$

Thus, by the definition of the isolated preferences,

$$(\hat{\eta}_t(i), \tilde{\eta}_{t+1}(i)) \succeq_i (\hat{\eta}_t(i), \hat{\eta}_{t+1}(i)).$$

Given that $\tilde{\eta}_t = \hat{\eta}_t$, the result above implies the desired result.

(ii) On the contrary, suppose that $\eta_t = \hat{\eta}_t$. By (i) of this lemma, reporting $\hat{P}_i(\hat{\eta}_{t-1})$ in period t and $P_i(\hat{\eta}_t)$ in period $t + 1$ is an also a profitable manipulation for agent i . Because $\eta_t = \hat{\eta}_t$, it must be that $P_i(\hat{\eta}_t) = P_i(\eta_t)$. Thus, at the new profitable manipulation i obtains $\eta(i)$ but this is what i obtains under truth-telling. Thus, we have reached a desired contradiction. The other part of (ii) follows immediately from the first part because $\eta_t \neq \hat{\eta}_t$ only if $P(\eta_t) \neq P(\hat{\eta}_t)$. \square

This lemma simplifies significantly our task of identifying the possible manipulations: any agent who is contemplating to manipulate the DA-IP mechanism should consider changing her isolated-preference report when she is one while submitting her true isolated-preference report when she is two. This means that if there are m schools, then we need to check at most $m!$ possible isolated-preference reports in order to figure out if an agent has a successful manipulation. This still is a daunting task, but the second part of the lemma offers a way to simplify our task further: we need to find all the period- t DA-IP matchings that correspond to some period- t isolated-preference report of i . As a first step in this direction, first we identify the schools that i can get matched in period t by reporting some isolated preference reports.

For the remainder of the current appendix we will only concentrate on period t . Thus, unless otherwise stated, we discuss only period- t DA-IP mechanism and period- t isolated-preference reports of the agents with respect to the previous period's DA-IP matching η_{t-1} . Thus, we simplify the notations by writing P_i instead of $P_i(\eta_{t-1})$.

We now prove Lemma 2 studied in the main text of the paper.

Proof of Lemma 2. To prove this lemma it suffices to show that the DA-IP mechanism allocates agent i to the attainable school listed highest in the submitted isolated-preference report of agent i . Let P_i^* be the submitted isolated-preference report of agent i . Let s^* be the highest ranked attainable school in i 's report. Contrary to the claim suppose that i is not allocated to s^* . Clearly, by the definition of the non-attainable schools, i cannot be allocated to any non-attainable school. Thus, i must be allocated to a attainable school that is listed after s^* . However, because s^* is a attainable school the DA-IP must allocate i to s^* for some report of i^* . As a result, if i 's true preferences were P_i^* , she would have had a successful manipulation of the DA-IP mechanism. This contradicts the strategy-proofness of the DA mechanism in static settings. \square

We here diverge somewhat from the material in Section 5 where we introduced a simpler version of our algorithm for simpler presentation. By definition, i can get allocated to any attainable school by submitting an appropriate isolated-preference report. However, there are some attainable schools that regardless of what school i is matched in period $t + 1$ are inferior to i 's DA-IP matching under truth telling, $\eta(i)$. Any period- t matching under which i is allocated to such a attainable school therefore can be ignored for our purposes of finding out whether i can manipulate the DA-IP mechanism.

Definition 13. *A school s is relevant if it is attainable and*

$$(s, s^*) \succ_i \eta(i)$$

where s^ is the school such that $(s^*, s^*) \succ_i (s', s')$ for all $s' \neq s^*$.*

We now look for ways to find all the DA-IP matchings under which i is allocated to some fixed relevant school s .

For each relevant school s and each set $S^s \subset S^{NA}$ we now split the non-attainable schools into two groups: redundant and non-redundant. If we set $S^s = \emptyset$, then the current definition of redundant school is equivalent to the one we used in Section 5.

Definition 14. *A school s' is redundant for (s, S') where s is attainable and $S' \subset S^{NA}$ if s' is attainable and if the DA-IP mechanism produces the same matching when i submits any two isolated-preference reports, P_i^s and \tilde{P}_i^s ,*

1. *that rank s as their highest attainable school*
2. *that the sets of schools ranked higher than s under P_i^s and \tilde{P}_i^s both contain S^s , and they differ only in that one under P_i^s does not contain s' while the one under \tilde{P}_i^s does.*

A school s' is non-redundant for (s, S') if s' is attainable and if it is non-redundant for (s, S') . We use the notations $S^R(s, S')$ and $S^{NR}(s, S')$ to denote the redundant and non-redundant schools for (s, S') , respectively.

Consider any $S' \subset S^{NA}$ and $S'' \subset S^{NA}$ such that $S' \subseteq S''$. From the definition above it is clear that

- (i) If $s' \in S^R(s, S')$, then $s' \in S^R(s, S'')$.
- (ii) If $s' \in S^{NR}(s, S'')$, then $s' \in S^R(s, S')$.

Fix a relevant school s and a subset of the non-attainable schools $S^s \subseteq S^{NA}$. We write \hat{P}_i^s to denote an isolated-preference report of i in which the set of schools ranked higher than s is S^s . We also write $\hat{\mu}_i^s$ to denote the DA-IP matching when i reports \hat{P}_i^s .

Lemma 8. *A school s' is redundant for (s, S^s) where s is relevant and $S^s \subseteq S^{NA}$ if and only if*

$$X_i^{s'}(\eta_{t-1}) < \min_{j \in \hat{\mu}_i^s(s')} \{X_j^{s'}\}(\eta_{t-1})$$

Proof. (a) We first prove the if part.

Fix any isolated-preference reports, P_i^s or \tilde{P}_i^s , such that

1. that rank s as their highest attainable school
2. that the sets of non-attainable schools ranked higher than s under P_i^s and \tilde{P}_i^s both contain S^s , and they differ only in that one under P_i^s does not contain s' while the one under \tilde{P}_i^s does.

We need to prove that the DA-IP mechanism produces the same matching if i submits either P_i^s or \tilde{P}_i^s .

Let μ_i^s and $\tilde{\mu}_i^s$ are the period- t DA-IP matchings corresponding when i submits P_i^s and \tilde{P}_i^s .

By Lemma 2, $\hat{\mu}_i^s(i) = \mu_i^s(i) = \tilde{\mu}_i^s(i)$. Now we define three static economies, \hat{E}_t , E_t , and \tilde{E}_t , in which the set of the agents in the three economies is the school-age agents in period t and the preference of each agent $j \neq i$ is $P_j \equiv P_j(\eta_{t-1})$. However, i 's preference is \hat{P}_i^s , P_i^s , and \tilde{P}_i^s in economies \hat{E}_t , E_t , and \tilde{E}_t , respectively.

Clearly, μ_i^s , $\hat{\mu}_i^s$ and $\tilde{\mu}_i^s$ are agents optimal or school worst stable matchings in economies E_t , \hat{E}_t , and \tilde{E}_t , respectively.

By Lemma 4, μ_i^s is stable in \hat{E}_t . Now because $\hat{\mu}_i^s$ is the school worst stable matching in economy \hat{E}_t and μ_i^s , we obtain that

$$\min_{j \in \hat{\mu}_i^s(s')} \{X_j^{s'}\}(\eta_{t-1}) \geq \min_{j \in \mu_i^s(s')} \{X_j^{s'}\}(\eta_{t-1}).$$

Combining condition above with the condition given in the lemma, we get

$$X_j^{s'} < \min_{j \in \mu_i^s(s')} \{X_j^{s'}(\eta_{t-1})\}. \quad (4)$$

By Lemma 4, $\tilde{\mu}_i^s$ is stable in E_t . Now because μ_i^s is the agents-optimal stable matching in economy E_t , every school-age agent j weakly prefers μ_i^s to $\tilde{\mu}_i^s$ in terms of her isolated preferences. If we show the opposite, i.e., every school-age agent j weakly prefers $\tilde{\mu}_i^s$ to μ_i^s in terms of her isolated preferences, then we are done. Thus, it suffices to show that μ_i^s is stable in economy \tilde{E}_t because matching $\tilde{\mu}_i^s$ is the agents optimal (static) stable matching in economy \tilde{E}_t . Suppose that μ_i^s is not stable in economy \tilde{E}_t . Given that the schools' priorities and each agent $j \neq i$'s preferences are the same in the two static economies, no agent $j \neq i$ and any school \tilde{s} can block μ_i^s in economy \tilde{E}_t because μ_i^s is stable in economy E_t . Therefore, there must exist \tilde{s} such that i and \tilde{s} block μ_i^s in economy \tilde{E}_t . Now recall that i is matched to s under both μ_i^s and $\tilde{\mu}_i^s$. Thus, to be a part of blocking pair \tilde{s} must be ranked higher than s in \tilde{P}_i^s . By the conditions given the lemma, if $\tilde{s} \neq s'$ then it is also ranked higher than s in P_i^s . Then, i and s' should have blocked μ_i^s in E_t , which would contradict that μ_i^s is stable in E_t . Thus, $\tilde{s} = s'$. However, due to (4) i and s' cannot block μ_i^s in economy \tilde{E}_t . Consequently, μ_i^s is stable in economy \tilde{E}_t .

(b) We now prove the only if part. Suppose that

$$X_i^{s'}(\eta_{t-1}) > \min_{j \in \hat{\mu}_i^s(s')} \{X_j^{s'}\}(\eta_{t-1}).$$

We now s' is non-redundant for (s, S^s) . On the contrary suppose that s' is redundant for (s, S^s) . Consider i 's isolated-preference report, \tilde{P}_i^s , in which the set of schools listed ahead of s is $S^s \cup s'$. We would reach a contradiction if we show that the DA-IP does not produce $\hat{\mu}_i^s$ when i reports \tilde{P}_i^s . Let \tilde{E}_t be a static economy in which the set of the agents is the school-age agents in period t , the preference of each agent $j \neq i$ is $P_j \equiv P_j(\eta_{t-1})$ and i 's preference is \tilde{P}_i^s . Given that the DA-IP produces a stable matching in \tilde{E}_t , all we need to show is that $\hat{\mu}_i^s$ is not static stable in \tilde{E}_t . This is clear because now i and s' will block $\hat{\mu}_i^s$ because i ranks s' ahead s and $X_i^{s'}(\eta_{t-1}) > \min_{j \in \hat{\mu}_i^s(s')} \{X_j^{s'}\}(\eta_{t-1})$. □

Now we are ready to present our algorithm to find all the period- t DA-IP matchings under which i is allocated to relevant school.

The Algorithm to Find the Set of Period- t Matchings under which i is Allocated to a Given Relevant School

Fix a relevant school s .

Round 0. Fix a isolated-preference report of i in which s is ranked first. Find the DA-IP matching when i submits this preference report and $\mathcal{M}_t^0(s)$ be the set that consists of this matching. Find all the redundant and non-redundant schools for (s, \emptyset) . Let $\mathcal{S}^0(s)$ be the set of all subsets of $S^R(s, \emptyset)$. Call $\mathcal{S}_1^0(s) = \{S' \in \mathcal{S}^0(s) : |S'| = 1\}$.

Round 1. For each $S' \in \mathcal{S}_1^0(s)$, fix an isolated-preference report of i in which s is the highest ranked attainable school and in which the set of schools that are ranked higher than s is S' . For all

these fixed reports of i , find the DA-IP matchings and denote the set of these matchings by $\mathcal{M}_t^1(s)$. If school $s' \in S^{NA}$ is redundant for s and some $S' \in \mathcal{S}_1^0(s)$, then we eliminate each $S'' \supseteq \{S' \cup s'\}$ from $\mathcal{S}^0(s)$. Specifically, update $\mathcal{S}^0(s)$ to $\mathcal{S}^1(s)$ as follows:

$$\mathcal{S}^1(s) = \mathcal{S}^0(s) \setminus \{S'' \in \mathcal{S}^0(s) : S'' \supseteq \{S' \cup s'\} \text{ for some } S' \in \mathcal{S}_1(s) \text{ \& } s' \in S^R(s, S')\}.$$

Set $\mathcal{S}_2^1(s) = \{S' \in \mathcal{S}^1(s) : |S'| = 2\}$.

Round k . For each $S' \in \mathcal{S}_k^{k-1}(s)$, fix an isolated-preference report of i in which s is the highest ranked attainable school and in which the set of schools that are ranked higher than s is S' . For all these fixed reports of i , find the DA-IP matchings and denote the set of these matchings by $\mathcal{M}_t^k(s)$. If school $s' \in S^{NA}$ is redundant for s and some $S' \in \mathcal{S}_k(s)$, then we eliminate each $S'' \supseteq \{S' \cup s'\}$ from $\mathcal{S}^{k-1}(s)$. Specifically, update $\mathcal{S}^{k-1}(s)$ to $\mathcal{S}^k(s)$ as follows:

$$\mathcal{S}^k(s) = \mathcal{S}^{k-1}(s) \setminus \left\{ S'' \in \mathcal{S}^{k-1}(s) : S'' \supseteq \{S' \cup s'\} \text{ for some } S' \in \mathcal{S}_k^{k-1}(s) \text{ \& } s' \in S^R(s, S') \right\}.$$

Set $\mathcal{S}_{k+1}^k(s) = \{S' \in \mathcal{S}^k(s) : |S'| = k + 1\}$.

The algorithm stops at the step \bar{k} where $\mathcal{S}_{\bar{k}+1}^{\bar{k}}(s) = \emptyset$ and the set of the matching $\mathcal{M}_t(s) = \bigcup_{k=0}^{\bar{k}} \mathcal{M}_t^k(s)$.

Proposition 3. *The algorithm above yields all the period- t matchings that are the result of the DA-IP mechanism for some report of i and in which i is allocated to s .*

Proof. Fix a relevant school s and period- t matching μ_t such that $\mu_t(i) = s$. In addition, suppose that the DA-IP mechanism produces μ_t when i submits an isolated-preference report, P_i^1 . Contrary to the proposition, suppose that μ_t is not found by our algorithm.

Let S^1 be the set of schools that i listed ahead of s at P_i^1 . By Lemma 4, we know that $S^1 \subseteq S^{NA}$. If $S^1 = \emptyset$, then we are done because μ_t must be found in round 0 of our algorithm. Thus, $S^1 \neq \emptyset$.

Suppose that each $s^1 \in S^1$ is non-redundant for $(s, S^1 \setminus s^1)$. This means that each $s^1 \in S^1$ is non-redundant for (s, S') where $S' \subseteq S^1 \setminus s^1$. Consequently, each set $S' \subseteq S^1$ is in $\mathcal{S}^{|S^1|-1}(s)$ (i.e., the set found in round $|S^1|$ of our algorithm). Furthermore, $S^1 \in \mathcal{S}_{|S^1|}^{|S^1|-1}(s)$. Thus, μ_t would have been found in round $|S^1|$ of our algorithm, which is a contradiction. Consequently, there exists $s^1 \in S^1$ such that s^1 is redundant for $(s, S^1 \setminus s^1)$.

Now fix one s^1 such that s^1 is redundant for $(s, S^1 \setminus s^1)$. Set $S^2 = S^1 \setminus s^1$. Consider i 's isolated-preference report in which the set of schools listed ahead of s is S^2 . If i submits this report then the DA-IP mechanism must produce μ_t , due to Lemma 8. Using similar arguments in the previous paragraph, we get that there exists $s^2 \in S^2$ such that s^2 is redundant for $(s, S^2 \setminus s^2)$. Using this argument repeatedly, we find sequences $\{s^1 \dots, s^{|S^1|}\}$ and $\{S^1 \dots, S^{|S^1|+1}\}$ such that $S^k = S^{k-1} \setminus s^k$ and s^k is redundant for (s, S^{k-1}) for all $k = 1, \dots, |S^1|$. Clearly, $S^{|S^1|+1} = \emptyset$. In addition, observe that if i submits an isolated-preference report in which the set of schools listed ahead of s is S^k where $k = 1, \dots, |S^1| + 1$, then the DA-IP produces μ_t . Because $S^{|S^1|+1} = \emptyset$, μ_t must be produced at round 0 of our algorithm. \square

Now we are finally ready to present our algorithm to check the manipulability of the DA-IP mechanism in a given economy by a given agent.

Fix an economy E and an agent i . Suppose that agent i is born in period $t \geq 1$.

The Algorithm to Check the Manipulability of the DA-IP Mechanism

Step 1. Run the DA-IP mechanism in economy E until period $t + 1$ and find the DA-IP matching of i . If i is matched to its most preferred school (\bar{s}, \bar{s}) then stop the algorithm. In this case, i cannot manipulate the DA-IP mechanism in economy E . If not move to the next step.

Step 2. Find the set of attainable schools of i in period t .

Step 3. Find the relevant schools of i in period t . If no relevant school exists then we stop the algorithm. In this case, i cannot manipulate the DA-IP mechanism in economy E . If a relevant school exists, then move to the next step.

Step 4. Consider the relevant schools sequentially. For a fixed relevant school s , find the set of period- t matchings, $\mathcal{M}_t(s)$, under which i is allocated to s . Consider each $\mu_t \in \mathcal{M}_t(s)$ sequentially and find the period $t + 1$ DA-IP matching, μ_{t+1} , assuming that every school-age agent j reports her period $t + 1$ isolated preferences as $P_j(\mu_t)$. If $\mu(i) \succ_i \eta(i)$, then stop the algorithm. In this case, i can manipulate the DA-IP mechanism. Otherwise, consider the next matching in $\mathcal{M}_t(s)$. If the algorithm does not stop before exhausting all the relevant schools and each period- t matchings under which i is matched to a relevant, then i cannot manipulate the DA-IP mechanism.