Imitation, Learning, and the Incomplete-Markets Selection Hypothesis

Andrea Giusto

Abstract

The market selection hypothesis states that markets work like biological systems that favor the survival of the fittest. This controversial proposition has been argued to hold in certain settings – Alchian (1950), Friedman (1953), Sandroni (2000), Blume and Easley (2006) – and not in others – De Long, Schleifer, Summers, and Waldmann (1990), Blume and Easley (1992), Blume and Easley (2006). In particular Blume and Easley (2006) give two examples of how this hypothesis does not hold in incomplete markets. This paper extends the analysis of the market selection hypothesis in incomplete market, by introducing rational imitation between traders. This paper shows that there exists situations in which diverse beliefs coexist, and the resulting equilibrium is different from the representative-agent Rational Expectation Equilibrium (REE). Furthermore this paper provides an evolutionary justification to the relationship between risk and returns, and it highlights that deviations from the REE hinge on flawed estimates of the risk parameter.

1 Introduction

Competitive market economies hinge crucially on the correct pricing of commodities. A recurrent theme in economics has been the idea that freely competitive markets are able to discover prices that drive a multitude of decision makers toward global economic efficiency. The theoretical results supporting this position are well understood, but recently the ability of real-world markets – especially financial markets – has been questioned by many within and without the profession. In truth, the evidence of extreme turbulence in the pricing of investment vehicles extends deeply in the past, as documented by Kindleberger (2000) for example. Financial instability represents an important conundrum in economics: even though casual observation suggests that the swings in the price of some financial assets are
often too volatile to reflect actual changes in fundamentals, the task of providing conclusive evidence of a bubble in a non-experimental market has proven itself to be elusive.

From a theoretical point of view, financial instability has provided a strong incentive to understand the conditions, if any, under which prices may deviate from what would be observed under the assumption of a representative agent endowed with rational expectations. A popular argument used to rule out the possibility of such deviations is the *market selection hypothesis*. Simply put, this hypothesis states that competitive markets automatically reward those that form the best forecasts at the expenses of those that form expectations in less efficient ways. Obviously, at the core of this argument there is a strong evolutionary component, in the sense it can be reconducted to the concept of the “survival of the fittest.”

De Long, Schleifer, Summers, and Waldmann (1990) show, by means of an example, that the evolutionary selection of rational expectations may not attain, under some restrictive conditions. They present a model in which non-rational traders may earn higher returns than the rational ones, thus weakening the original evolutionary argument in favor of the market efficiency hypothesis. The main limitation of the De Long et al. model of noise trading is the fact that the dynamics of the population of investors are derived under the assumption that, in each period, the measure of investors that change their trading strategy from “rational” to “noise” limits to zero. Blume and Easley (1992) extend the analysis to a general equilibrium situation and find that rational investors are not necessarily selected for in the marketplace. Sandroni (2000) analyses the evolutionary selection problem with market completeness and endogenous saving schedules, and finds that the markets select those traders that behave as if they had rational expectations. Blume and Easley (2006) provide an explanation of the result of Sandroni (2000) in terms of Pareto-efficiency of the trading allocation. Furthermore, Blume and Easley (2006) remove the assumption of market completeness and study some examples in which the market selection hypothesis may fail to attain in the case of incomplete markets.

In this paper, I extend the analysis of the market selection hypothesis when markets are incomplete. The key innovation introduced in this paper are imitation between traders and learning, and I ask whether the market selection hypothesis is valid under an optimal learning rule. More specifically, I consider finitely-lived subjective-expected-utility maxi-

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1 See for example Alchian (1950), and Friedman (1953).
mizers that face the problem of optimally choosing among several alternative models to forecast future prices. The forecasting problem is complicated by fact that agents have finite lives which means that the return to an investment equals the sum of dividends and capital gains. Since the price at the moment of the investment liquidation depends on how wealth is distributed across the various belief systems that are present in the market, the agent’s problem is to select a model that captures how changes in distribution of beliefs will impact the selling prices. Markets are incomplete here because there is no asset to insure against adverse price movements. The results contained in this paper offer two main insights on the behavior of this type of incomplete asset markets: first, it is shown that the market selects against certain belief systems, to favor others that satisfy an intuitively appealing consistency requirement. In particular, only those belief that provide balance between risks and returns survive in the market and in this sense, this result provides an evolutionary justification to a fundamental concept of the asset pricing literature.

Second, I show that the representative-agent rational-expectations equilibrium (REE) is one possible equilibrium situation for incomplete-market prices under imitation and learning. Nevertheless, in this model, there are other instances of heterogeneous-beliefs equilibria in which the beliefs active on the market interact with each other in a way that both support each other and move the price away from fundamentals. In the following I will refer to this of situation as heterogeneous-beliefs equilibria (HBE). An HBE can be seen as a specific instance of a misspecification equilibrium of the type outlined in Branch and Evans (2006), since here the agents are assumed to be boundedly rational in the sense that they are limited in type and number of models that they can use to form expectations. Interestingly, at an REE, the agents trading on the basis of REE-beliefs are selected against, and driven away from the market. This particular feature of these kind of equilibria suggests an original and intriguing interpretation of the episodes of financial instability as a “physiological” response of a market stuck in an HBE in order to restore fundamental pricing. In fact, if a direct transition from an HBE to the representative-agent REE is not possible because REE-traders are driven away from the market, it is possible nevertheless that there is a path from an HBE to the fundamental pricing equilibrium. This systematic exploration of this issue is left for future work.

Differently from De Long et al. (1990), noise traders are replaced here by a lack of
Agents are aware that other agents may have different beliefs, but they do not have information on how wealth is distributed across belief systems, other than the market price observed at a given instant. Accordingly, a coordination around the REE does not attain automatically, and the possibility of deviations from fundamentals remains open. Furthermore, I allow the flows of traders among belief systems to be endogenously determined, and in particular they will be proportional to the relative performance of each competing belief system at any given instant. The importance of having endogenous flows between traders sub-populations is clear in light of the results of Brock and Hommes (1998) and Branch and McGough (2008) which use numerical methods to show that the qualitative aspects of the dynamics of the asset prices are determined by the intensity of imitation and by the assumptions on the number, type and numerosity of different groups of noise traders. Both papers’ numerical simulations show that the price dynamics of an asset that under rational expectations is constantly priced to unity, can be incredibly complex under different imitation intensities, reaching chaotic complexity.

The model presented in this paper is an application of the overlapping-generations model presented in Weibull (1995, Chapter 4.) The rationale for this modeling choice is the availability of a number of theoretical results that link the stationary states of the population dynamics to the set of Nash equilibria of particular games whose payoff functions are connected to the growth rates of the various sub-population shares. More specifically, at each point in time, the state-space consists of the distribution of agents across a finite number of “expectational models” which in turn are responsible for shaping the individual demand schedules. A Walrasian auctioneer collects the agents demands and announces the clearing price, which is in turn used by the agents to select an expectational model. A recursive specification for the population dynamics is derived under specific assumptions about the expectational models and the model selection mechanism; given this formulation a dual problem is studied by employing game theoretic equilibrium concepts to identify the stationary states of the population of investors.

By intensity of imitation it is intended here the share of investors that, at each point in time review their investment strategy and imitate the best alternative. In De Long, Shleifer, Summers, and Waldman (1990) this share is assumed to limit to zero.
2 The Model

Consider a simple overlapping generations model in which agents live for two periods, have CARA utility, and face a portfolio selection problem as in De Long, Schleifer, Summers, Waldman (1990). There are two available assets: first, the safe asset \( s \) that offers a return equal to \( r \) per period, and second, the risky asset \( u \) that offers a return of \( r + \varepsilon \) per period where \( \varepsilon \) is a normally distributed random variable with zero mean and variance \( \sigma_\varepsilon^2 \).

The price of the asset \( s \) is normalized to one in every period, and the price of \( u \) at time \( t \) will be denoted \( p_t \). The existence of a unique, correct pricing model for the asset \( u \) is not common knowledge among the agents. Therefore, there are two distinct sources of risk involved when trading \( u \): on top of the dividend risk, an investor faces also the risk that the trading activity of other investors causes movements in the price of the asset thus changing the actual returns via capital gains or losses. Clearly, the overlapping-generations-structure and the assumption that agents have a finite planning horizon are key here; in fact, if agents had infinite lives, they may not be concerned with the price movements of the risky assets, as they would not be forced to liquidate the asset in any given period.

Belief systems concern the distribution of the price of the risky asset and, for analytical tractability the beliefs will be assumed to be of the normal family. Let us denote with \( K \) the set of the belief systems, \( K \equiv \{(p^k, \sigma_k^2) \in \mathbb{R}_+^2 \}^{K}_{k=0} \). An agent belonging to category \( k \in K \), reckons that the distribution of the price of the risky asset is normally distributed with mean \( p^k \) and variance \( \sigma_k^2 \). In this respect, agents are assumed to be boundedly rational, since they are not able to chose non-normal beliefs even though in general, a different distribution is likely to provide better forecasts. Furthermore agents rationality is bounded further by the finiteness of the set \( K \). To make the comparison to the case of rational expectations easier, it is convenient to redefine the expectations of each type \( k \) in terms of deviations from \( p^R \), the price that would be expected if there was common knowledge of a unique REE. Therefore, defining \( \rho_k = p^k - p^R \), an equivalent specification of the set of the expectational models is \( K \equiv \{(p^R + \rho_k, \sigma_k^2) \in \mathbb{R}_+^2 \}^{K}_{k=0} \), with \( \rho_k \in \mathbb{R} \) for all \( k \). A \( k \)-type expects the price of the risky asset to be equal to the rational expectations price \( p^R \) plus \( \rho_k \). Assume that the belief system with index zero (\( k = 0 \)) is the REE belief system (\( \rho^0 = 0, \sigma_0 = \sigma_\varepsilon \)). This is the only restriction imposed on the set of beliefs, other than normality and finiteness.
At birth, agents are endowed with beliefs from the parent agent, a unit of wealth, and a utility function of the form
\[ U = -e^{-\frac{1}{2} \gamma W_o}, \]
where \( \gamma \) is the risk aversion parameter and \( W_o \) is the total wealth accumulated by the agent at the end of his/her young stage. The agent’s problem is the following,
\[ \max_{\lambda^k} \lambda^k (p^k + r) + (1 - \lambda^k p_t)(1 + r) - \frac{1}{2} \gamma (\lambda^k \sigma_k)^2, \]
where \( \lambda^k \) is the quantity of asset \( u \) demanded by an investor that belongs to category \( k \).

The necessary and sufficient first order condition yields the demand function for an agent of type \( k \) at time \( t \),
\[ \lambda^k(p_t) = \frac{p^R + p_k + r - p_t(1 + r)}{\gamma \sigma_k^2}. \]

Let \( x^k_t \) for \( k \in K \) denote the proportion of investors of type \( k \) present in the population at time \( t \), and let \( x_t = (x^1_t, \ldots, x^K_t) \) denote the state of the population at time \( t \), where \( \sum_{k \in K} x^k_t = 1 \). Normalizing the supply of \( u \) to unity as well, it is possible to solve for the market price at time \( t \) as a function of the contemporaneous population composition \( x_t \) by solving the market clearing condition \( (1 = \sum_{k \in K} \lambda^k x^k_t) \) for \( p_t \):
\[ p(x_t) = \frac{1}{1 + r} \left[ p^R + r + \bar{\rho}(x_t) - \gamma \bar{\sigma}(x_t) \right] \quad (1) \]
where \( \bar{\rho}(x_t) = (\sum_{k \in K} \frac{p_k x^k_t}{\sigma_k^2})/(\sum_{k \in K} x^k_t \sigma_k^2) \) represents a weighted average of the population’s distribution of opinions about the correct price of the risky asset, with weights equal to the population share in each category, divided by the risk perceived by each category. The weighting scheme has an appealing intuitive explanation: the more popular is a belief in the population, the more the price will be influenced by it; but at the same time, the higher the risk perceived by a certain class of investors the smaller their positions will be in the market, and therefore the lesser the effect on the price of their opinions. The term \( \bar{\sigma}(x_t) = 1/(\sum_{k \in K} x^k_t \sigma_k^2) \) measures the average perceived risk involved in the trading of the risky asset, weighted by the the subpopulations shares, and it represents a risk premium. As equation (1) shows, the price of \( u \) depends exclusively – although non linearly – on the population state at time \( t \) and on the parameters of the model. The market clearing condition alone is enough to guarantee that the market acts as an *aggregator* of beliefs:
2 The Model

net of risk aversion and discounting, the price of the risky asset reflects the distribution of
different opinions present in the population of traders in each time period. The question
of whether the market can act as a “superaggregator” of beliefs as well – in the sense that
it is able to select the “best” beliefs – is the main object of investigation of the rest of this
paper.

2.1 Rational Expectations Equilibrium

It is useful to derive the pricing the market determines when the existence of a unique
correct model with variance $\sigma^2_\epsilon$ is assumed to be common knowledge. When the population
state is $(x_0 = 1, \ldots, x_K = 0)$ the agents’ uncertainty caused by the dispersion of opinions
in the population is zero, and therefore demand simplifies to

$$\lambda_t^R = \frac{p^R + r - p_t(1 + r)}{\gamma \sigma^2_\epsilon}$$

The market clearing condition implies that the expected price is equal to

$$p^R = 1 - \frac{\gamma}{r} \sigma^2_\epsilon$$

at all times. The corresponding population dynamics are simply described by the state
$x^0_t = 1$ at all times $t$, while the price dynamics are stationary and they are described by a
constant plus IID Gaussian noise $(p^R + \epsilon_t)$.

2.2 Imitation and Learning

After the investment decision is made and the Walrasian auctioneer announces the clearing
price, investors have a chance to evaluate their beliefs by direct comparison with some
alternative. In particular, I will assume that each agent receives (noisy) information about
another investor randomly sampled from the population. The random sampling of beliefs
can be regarded as simplistic, in the sense that it abstracts from the differential incentives
that different traders have to collect information.\textsuperscript{3} Nevertheless this simple assumption

\textsuperscript{3} For example, if the processing of information is somehow costly, an agent that has beliefs that lead
him to invest heavily in the risky asset should optimally spend more time and effort than someone that has
beliefs that keep her away from $u$. Random sampling cannot account for this dimension of the problem.
captures in a tractable way the multitude of communication channels between investors that characterize real-world financial markets.

The decision to change one’s beliefs is based on the relative likelihood of the last price observed in the market. The use of the past data on prices is not appropriate because, in general, the series of prices is not guaranteed to be stationary, since the population shares evolve over time. Therefore the agents will compare the likelihood of observing \( p_t \) according to each beliefs system and pick whichever distribution appears to better describe the data. If agents were allowed to use flexible distributional forms, then they would behave as Bayesian econometricians. Although more appealing from a conceptual point of view, allowing for Bayesian traders would require the loss of analytical tractability. The noise introduced in the imitation step is both useful from the modeling perspective, and it also constitutes a device to account for idiosyncratic differences of agents over beliefs, and communication imperfections. Whatever the agent’s choice, the corresponding belief system is passed on to the offspring.

To fix ideas it is useful to consider first the case of a \( k \)-type agent that samples a \( j \)-type without the introduction of noise. Under the current assumptions, the \( k \)-agent will compare the likelihood of the observed price \( p_t \) under his original system of beliefs to the same quantity under the competing beliefs \( j \). Because all the random variables involved in this comparison are assumed to be normal, the decision rule based on the likelihood ratio test can be written as follows

\[
\frac{L(p_t|\rho_k, \sigma_k)}{L(p_t|\rho_j, \sigma_j)} < 1 \implies j \succ k
\]

where \( L(\cdot|\rho, \sigma) \) is the normal probability density function with the mean \( \rho \) and variance \( \sigma^2 \), and the symbol \( \succ \) denotes a preference relation. An equivalent and simpler comparison rule is obtained by considering the quantity \( S_i = \frac{p_t - p_i}{\sigma_i} \) for \( i = j, k \). Letting \( \mathcal{L} \) denote the standard normal pdf, an alternative but equivalent decision rule is

\[
\begin{cases}
\sigma_j < \sigma_k \implies j \succ k, & S_j = S_k = 0 \\
\frac{\mathcal{L}(S_{k,t})}{\mathcal{L}(S_{j,t})} = \exp \left( \frac{1}{2} S_{j,t}^2 - \frac{1}{2} S_{k,t}^2 \right) < 1 \implies j \succ k, & \text{otherwise}
\end{cases}
\]
or alternatively

\[
\begin{align*}
\sigma_j < \sigma_k \Rightarrow j > k, & \quad S_j = S_k = 0, \\
-S_{j,t}^2 > -S_{k,t}^2 \Rightarrow j > k, & \text{ otherwise.}
\end{align*}
\]

This rule highlights that, for a given belief system \( j \in K \) an appropriate “fitness” measure is \(-S_{j,t}^2\) – except for the non-generic case \( S_i = S_j = 0 \). The seemingly odd choice of the letter \( S \) to denote a \( t \)-statistic is motivated by the similarity of the key quantities in (3) with the Sharpe ratio, commonly used by practitioners to rank assets in the financial markets.\footnote{The Sharpe ratio is defined as the extra return per unit of risk that an asset offers over the risk-free asset. In this sense the quantity \( S \) is not properly a Sharpe ratio because it is defined in levels rather than changes.} For technical reasons that will be clear shortly, it will be necessary to have a positive measure of fitness to insure proper population dynamics. Therefore let the fitness measure be \( \mu_k^i = \kappa - S_{k,t}^2 \), for \( k \in K \), where \( \kappa \) is a positive constant chosen to insure that the worst performing belief system at any time has nevertheless positive fitness measure; formally, \( \kappa - S_{i,t}^2 > 0 \) for \((i, t) = \arg\min_j \tau - S_{j,\tau}^2\). The parameter \( \kappa \) may be thought of as an exogenous “birth” rate for each category of agents, so that in every period each belief is represented in the market.

Introducing noise in the imitation process, leads to the following decision rule

\[
\text{if } \frac{\mu_j}{\mu_k} > \zeta \text{ then system } j \text{ is superior}
\]

where \( \zeta \) is a uniformly distributed positive real random variable, with symmetric support around 1. To reiterate, the variable \( \zeta \) is meant to account for the possibility of communication imperfections and/or idiosyncratic preferences for belief systems. Denoting the cumulative density function of \( \zeta \) with \( \Phi \), it is possible to write the probability that a \( q \)-investor will adopt beliefs with index \( i \) (conditional on sampling someone with \( i \)-beliefs) as follows

\[
\text{Prob}_{qi} = \text{Prob}(\zeta < \frac{\mu_i}{\mu_q}) = \Phi \left( \frac{\mu_i}{\mu_q} \right)
\]

and, denoting with \( n_t \) and \( n_i^t \) the (Lebesgue) measure of the general population of agents and of the population of type \( k \) respectively, the unconditional probability that a \( q \)-type
will adopt beliefs with index $i$ is,

$$Prob_i^q = \frac{n_i^q}{n_t} \Phi\left(\frac{\mu_i}{\mu_q}\right) \quad (4)$$

By force of continuity and the laws of large numbers the stochastic process governed by the probabilities (4) can be characterized with a system of deterministic difference equation. For given $k \in K$, the change in the measure traders with $k$-beliefs between any two time periods is equal to the inflow minus the outflow of traders into and from $k$. As the inflow into $k$ is equal to $\sum_{j \neq k} n_i^j Prob_j^k$ while the outflow is equal to $\sum_{j \neq k} n_i^k Prob_j^k$, it turns out that

$$n_{t+1}^k - n_t^k = \sum_{j \in K} n_i^j Prob_j^k - \sum_{j \in K} n_i^k Prob_j^k$$

and using (4) the total measure of agents of type $k$ at time $t + 1$ is

$$n_{t+1}^k = \sum_{j \in K} \frac{n_i^j n_i^k}{n_t} \Phi\left(\frac{\mu_k}{\mu_j}\right) \quad (5)$$

Exploiting the linearity of $\Phi$ it is possible to rearrange equation (5) as follows

$$n_{t+1}^k = n_t^k \mu_t \Phi\left(\sum_{j \in K} \frac{n_i^j}{n_t} \frac{1}{\mu_j}\right) \quad (6)$$

The quantity in parenthesis is a weighted average of the reciprocal of the fitness measure, with weights equal to the population shares $x_i^j = \frac{n_i^j}{n_t}$; this quantity will be denoted with $\mu_t^{-1}$ in the following. Summing over $k$ in the last expression yields

$$n_{t+1} = n_t \mu_t \Phi(\mu_t^{-1}) \quad (7)$$

where $\mu_t = \sum_{k \in K} \frac{n_i^k}{n_t^k} \mu^k_t$ represents the (weighted) average population fitness. Dividing (6) by (7) finally yields the system of difference equations that govern the population dynamics,

$$x_{t+1}^k = \frac{\mu_t^k}{\mu_t} x_t^k, \quad k \in K \quad (8)$$
Equation (8) is the discrete-time version of the familiar replicator dynamics, which clearly shows the necessity of positive fitness measures, since negative population shares do not make sense.

In general, without further detail on the investors’ categories it is not possible to study the dynamic system (8). To overcome this difficulty, I will rely on a number of evolutionary game-theoretic results that link the stable states of specific classes of evolutionary dynamics to the set of (perfect) Nash equilibria of an underlying game. It is important to mention here that the specific form of the game being analyzed may very well not have an intuitively appealing explanation, but nevertheless, this is not a problem since the game in question is merely instrumental to the identification of the asymptotically stable states of the particular population dynamics justified this far and summarized in equation (8).

Notice that there is no room for the creation of new beliefs in this setup since the process of reproduction is free from mutations. Nevertheless, it is possible to deal (locally at least) with the problem of mutations through focusing on population states that are asymptotically stable. In this way, one can address concerns about mutations that leave the system inside a neighborhood of an equilibrium point. In fact, a small enough mutation “displaces” the system from an asymptotically stable state to some state contained in its basin of attraction.

3 The Dual Problem

The system of equations (8) is hard to study in this form because the fitness measure for a specific sub-population depends on the entire distribution of agents over the beliefs vector. In fact the μ’s depend on the price \( p_t \) which equation (1) shows to be a function of the population state at time \( t \). By rewriting the explicit dependence of (8) on \( x_t \) one obtains

\[
x_{t+1}^k = \frac{\kappa - \left( \frac{p(x_0^t,...,x^K_t)-p^k}{\sigma_k} \right)^2}{\kappa - \sum_{j \in K} x_j^t \left( \frac{p(x_0^t,...,x^K_t)-p^j}{\sigma_j} \right)^2} x_t^k, \quad k \in K
\]

\(^5\) A possible way to introduce mutations would require the use of numerical methods such as genetic algorithms, for example.
where \( p \) is defined in equation (1). The task of identifying analytically the rest points of this system of equations seems prohibitive but nevertheless it is possible through the application of a series of results from the literature on evolutionary game-theory. These results link the stationary states of the replicator dynamics to the Nash equilibria of certain games.

Let \( \Gamma \) denote a \( 2 \times 2 \) game in normal form. The set of available strategies is \( K \), a mixed strategy on \( K \) is denoted with \( \chi = (\chi_0, \ldots, \chi_K) \) and \( e^i \) denotes the \( i \)-th vertex of the \( K \)-dimensional simplex \( \Delta \). The payoff to a player playing pure strategy \( i \) is \( u(e^i, \chi) = \kappa - S_i^2(\chi) \) where \( S_i^2(\chi) \) is defined as in the previous section. As the following proposition shows, the Nash equilibria of this game are closely related to the stationary states of the dynamics of interest.

**Proposition 1.** The set of symmetric Nash Equilibria of \( \Gamma \) is a subset of the stationary states of the population dynamics (8).

**Proof:** let \( \Delta \) denote the \( K \) dimensional simplex, and let the set of stationary states of the dynamics (8) be denoted \( \Delta^0 \). Formally

\[
\Delta^0 = \{ \chi \in \Delta : \kappa - S_{k,t}^2 = \kappa - \bar{S}_t^2, \forall k \in C(\chi) \}
\]

where \( C(\chi) \) is the support of the probability distribution \( \chi, C(\chi) = \{ i \in K : \chi_i > 0 \} \). The set of symmetric Nash Equilibria of \( \Gamma \) is

\[
\Delta^{NE} = \left\{ \chi \in \Delta : \kappa - S_{j,t}^2 = \max_{z \in \Delta} \kappa - \sum_{i \in K} z_i S_i^2(\chi), \forall j \in C(\chi) \right\}
\]

but if strategy \( j \) satisfies the condition to belong to a mixed strategy profile that is a symmetric Nash equilibrium, then this strategy must earn the maximal payoff, which must be equal to the average payoff \( \kappa - \sum_{j \in K} \chi_j S_j^2(\chi) = \kappa - \bar{S}^2 \). This implies \( \Delta^{NE} \subset \Delta^0 \).

The proof of Proposition 1 is a straightforward application of the arguments contained in Weibull (1995, Chapter 3). Nevertheless the proof needs to be written explicitly because
Weibull assumes that the payoff function is linear function in the population shares, and this is not satisfied in the present case.

### 3.1 Best Replies

As customary, I will use the best reply correspondence to find the equilibria of $\Gamma$ by identifying those mixed strategies that are best replies to themselves. The formal equivalence between a mixed strategy in $\Gamma$ and a population state in the model of section 2.2 will then allow the application of Proposition 1 for the purpose of finding some of the stationary states of the evolutionary dynamics of the market considered here.

Since the trading model does not allow traders to “average” different beliefs (which would correspond in $\Gamma$ to a mixed strategy), but only the markets can behave in this way, I will restrict the focus to pure best-reply correspondences. In this case, game $\Gamma$ presents a particularly simple strategic interaction: simply put, the payoff of a player that can choose only pure strategies is maximized when he or she selects the strategy that implies an expected price as close as possible to the observed market price. To see this point it is useful to define the set of beliefs $\tilde{K} \equiv (c \in K, p_c = p(x))$ explicitly write the best-reply correspondence,

$$
\beta(x) = \begin{cases} 
  k = \arg \min_{c \in \tilde{K}} \sigma_c^2, & \text{if } \tilde{K} \neq \emptyset \\
  k = \arg \min_{c \in K} \left( \frac{p_c - p(x)}{\sigma_c} \right)^2, & \text{otherwise}
\end{cases}
$$

As (9) shows, the best pure strategies are the ones that come as close as possible to the price observed on the market, and for the case in which two or more belief systems have the expected price level exactly equal to the observed price, then the beliefs with lower variation is the best reply. It is now possible to use (9) to identify the equilibria of $\Gamma$.

### 3.2 Consensus Equilibria

In this section I consider the pure strategy Nash equilibria of $\Gamma$ which, in the trading model, represent those situations in which all the traders in the market adopt a single expectational model. The following proposition shows that not just any strategy may be selected in this game to be a “consensus equilibrium.”
Proposition 2. Let $K_c \subset K$ be defined as follows

$$K_c \equiv \left\{ k \in K : \rho_k = \frac{\gamma}{r} (\sigma^2 - \sigma^2_k) \right\}$$

(10)

Only the beliefs that belong to $K_c$ are possible candidates for a consensus equilibrium.

proof: fix $k \in K$ and let $x_k = 1$. Taking the first order condition with respect to $p_c$ for the minimum problem in equation (9) yields $p_c = p(x)$. When $x_k = 1$ the function $p(e^k)$ equals $\frac{1}{1+r} [p^R + r + \rho^k - \frac{\gamma}{\sigma^2_k}]$. Therefore for $k$ to be a best reply to itself the following condition must be met

$$p_k = \frac{1}{1+r} \left[ p^R + r + \rho^k - \gamma\sigma^2_k \right]$$

Rearranging yields the condition in (10).

In terms of the dynamics of the trading model, i.e. equations (8), any consensus situation can be forced into a rest point by simply starting the system with $x_k = 1$ for some $k \in K$. In such a case, it is easy to see that the growth rates of the populations different from $k$ are zero at all times. This is an “artifact” of the replicator dynamics, but what is different about the strategies in $K_c$ from a trivially-stable situation is that, depending on initial conditions involving more than one strategy, there is the potential for the dynamics to converge to a point in $K_c$, and this is not in general true for states not contained in $K_c$.

Proposition 2 shows that only those beliefs that present a balance between the risk and the extra returns of the risky asset are potential candidates for an equilibrium situation in which there is agreement about the “correct” pricing model. An important point shown by proposition 2 is that the price in such an equilibrium depends only on the risk perceived by the market about the returns that the asset $u$ is able to offer. In a stationary situation, agents can use data about the past returns of $u$ and obtain a consistent estimate of $\sigma \varepsilon$ thus forcing the market to the representative-agent rational expectation equilibrium. But if the data generating process is non-stationary, as it may be here, any price movement could be interpreted as either a product of an idiosyncratic shock, or to a change in the population composition, so that it is not obvious that agents would be able to unmistakably
estimate the variance of the returns to the risky asset. The convergence to an REE, therefore, depends here exclusively on the information about the risk, and whether or not real world traders have good estimates of the risks involved in trading financial assets is ultimately an empirical question. In support of the view that flawed perceptions of risks may be responsible of markets failures, it can be argued that in the real world much of the variability of the dividend series is actively smoothed out by companies, thus making the estimation problem harder.

3.3 Heterogeneous Beliefs Equilibria

In this section I will show, by means of an example, that this model admits more types of equilibria than the consensus ones. In particular, it is possible to show that depending on the composition of beliefs in the market, the price may permanently deviate from $p^R$. This point is easily illustrated in Figure 3.3 where it is shown a situation in which a certain proportion of the population have beliefs denoted by 1 and remainder believe in distribution 2. In this situation, when distribution of the types 1 and 2 is such that the clearing price is equal to $\bar{p}$ then it is easy to see that a Nash equilibrium for the game $\Gamma$ is attained for that particular mixed strategy that corresponds to the distribution of beliefs between 1 and 2, and therefore the price dynamics will be centered around $\bar{p}$. In this case, a fundamental-pricing trader (with beliefs denoted by the dashed line) not only would make poor financial decisions (he would underinvest in the risky asset), but also, given the extremely low likelihood of his beliefs, he would not be able to convince any other agent to adopt his own beliefs. Furthermore, at the end of his old stage of life, this trader would almost surely pick any of the other two belief systems to pass on to his offspring. The only chance for the REE belief system to gain a following is in the exogenous dividend process: if a particularly bad occurrence of the dividend happened to depress the price down below
a critical threshold, then the dotted beliefs would look more reasonable and they would take over the market.

**Example 1.** Consider the following simple case in which the set $K$ contains only three strategies:

$$K = \left\{ k^0 : \left(1 - \frac{\gamma}{r} \sigma^2, \sigma^2 \right), k^1 : \left(1 - \frac{\gamma}{4r} \sigma^2, \frac{\sigma^2}{4} \right), k^2 : \left(1 - \frac{\gamma}{16r} \sigma^2, \frac{\sigma^2}{16} \right) \right\}$$

Each element of $K$ specifies here the mean and variance of a normal distribution, so that for example according to beliefs $k^0$ the price is normally distributed with mean $1 - \frac{\gamma}{r} \sigma^2$ and variance $\sigma^2$. Consistently with the previous assumptions the element of $K$ with index 0 corresponds to the REE beliefs. Each belief populating $K$ represents a possible consensus equilibrium as it is easily demonstrated by showing that both beliefs satisfy condition (10). Accordingly, each of the beliefs of this example are “rationalizable” in the sense that, contingent on a particular estimate of the risk of $u$, agents that understand proposition 2 and that believe that a consensus will be reached eventually, have mean expectations that are consistent with a long run equilibrium of this model. But the convergence to a consensus is not a necessary outcome of the population dynamics of this model! In fact, there is a stable population state at the simplex point $\bar{x} = (x^0 = 0, x^1 = \lambda, x^2 = 1 - \lambda)$, where $\lambda = \frac{2(1 + r)}{12 + 3(1 + r)}$, which lies between 0 and 1 for $r \in [0, 17]$. That this point is an equilibrium is easily demonstrated by determining the clearing price resulting from the population state $\bar{x}$ and by showing that both $k^1$ and $k^2$ earn the maximal payoffs in the game $\Gamma$. From equation (1) it is possible to see that one needs to calculate $\rho^1 = 1 - \frac{\gamma}{4r} \sigma^2 - p^R = \frac{3\gamma}{16r} \sigma^2$ and $\rho^2 = 1 - \frac{\gamma}{16r} \sigma^2 - p^R = \frac{15\gamma}{16r} \sigma^2 / 4$. It follows that

$$\bar{\rho}(\bar{x}) = \frac{3\gamma \sigma^2 / 4r}{\sigma^2 / 4} + \frac{15(1 - \lambda) \gamma \sigma^2 / 16r}{\sigma^2 / 16} = \frac{3\gamma}{16} (15 - 18\lambda) \frac{\sigma^2}{16 - 12\lambda}$$

$$\bar{\sigma}(\bar{x}) = \frac{\lambda}{\sigma^2 / 4} + \frac{1 - \lambda}{\sigma^2 / 16} = \frac{\sigma^2}{16 - 12\lambda}$$

and

$$p(\bar{x}) = \frac{1}{1 + r} \left[ 1 - \frac{\gamma}{r} \sigma^2 + r + \frac{3\gamma}{16 - 12\lambda} \sigma^2 - \frac{\gamma \sigma^2}{16 - 12\lambda} \right]$$
substituting for $\lambda$ yields $p(\bar{x}) = 1 - \frac{\gamma}{8r}\sigma_\varepsilon^2$. At this price both systems of beliefs $k^1$ and $k^2$ possess a fitness measure equal to

$$\mu^1 = \mu^2 = \kappa - \frac{1}{16} \left( \frac{\gamma}{r}\sigma_\varepsilon \right)^2$$

while the REE strategy performs a lot worse

$$\mu^0 = \kappa - \frac{49}{64} \left( \frac{\gamma}{r}\sigma_\varepsilon \right)^2$$

which shows that when the population state is $\bar{x}$ the REE trader would be selected against in this market.

\[\square\]

4 Conclusion

The nature and characteristics of any heterogeneous belief equilibrium necessarily depends on the particular models active on the market and how they confirm each other producing deviations from fundamental prices. The result on the existence of HBEs lacks specificity, in the sense that pretty much any price can be designed to be an evolutionary equilibrium, by opportunely populating the set $K$. Nevertheless, HBEs clearly demonstrate the inappropriateness of assuming a representative agent, because a situation like the one pictured in Figure 3.3 cannot even be conjectured under the fiction that the market participants all share a common set of beliefs. Additionally, in Figure 3.3 and in example 1, only two set of beliefs are considered for simplicity, but in principle it is possible to establish the result with an arbitrary number of beliefs coexisting in the market.

The model considered here features boundedly rational agents that lack information about the distribution of beliefs in the population of traders active on the market for the asset $u$. It is possible and interesting to extend the analysis conducted this far by employing numerical methods and relaxing the assumption of normality in favor of more flexible functional forms, to explore the importance of bounded rationality relative to the informational constraints. This exercise is left for future work on this topic.

It is also possible by using numerical methods to study which kind of price dynamics
this model is capable of generating and in particular whether this framework is capable of replicating the typical features of bubbles, such as the prolonged and accelerating build up phase followed by sudden declines in the value of the assets. In particular, it would be desirable to understand under which parameterizations of the set $K$ such occurrences may – if at all – happen. This investigation is left for future work.

Another area of potentially fruitful further study of this model is highlighted by example 1. There it is clear that the HBE that can produce persistent price deviations from fundamentals attains at a particular population composition that is *not* independent of the the rate of return of the riskless asset $r$. It follows that a policymaker that has control over the parameter $r$, may disrupt the HBE to try and induce a pricing closer to fundamentals. This investigation is, again, left for future research.

References


