Lower bounds from tile covers for the channel assignment problem

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Lower Bounds from Tile Covers for the Channel Assignment Problem *

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Abstract

A method to generate lower bounds for the channel assignment problem is given. The method is based on the reduction of the channel assignment problem to a problem of covering the demand in a cellular network by pre-assigned blocks of cells, called tiles. This tile cover approach is applied to networks with a co-site constraint and two different constraints between cells. A complete family of lower bounds is obtained which include a number of new bounds, and improve or include almost all known clique bounds. When applied to an example from the literature, the new bounds give better results.

1 Introduction

Finding an optimal assignment of communication channels in a cellular network is a difficult combinatorial optimization problem which has received considerable attention over the last decade. This is due to the explosive growth of wireless communications and the scarcity of the radio spectrum. The channel assignment problem (CAP) is NP-complete even in a drastically simplified form, and, consequently, most efforts have gone towards the development of good heuristics. (Recently, integer programming techniques which can lead to exact solutions have been used. See, for example, [11]). Lower bounds play an important role in the evaluation of any heuristic or approximation algorithm. Moreover, lower bounds can help to identify the structures that form the bottleneck for a particular instance, and this information can, in turn, be used to find better assignments.

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A basic model for a cellular network describes it in terms of the *demand* for channels in each cell and a set of *separation constraints* which prescribe minimal separations that must exist between channels assigned to certain cells in order to avoid interference. The goal of the CAP is to assign channels (represented by integers) to the cells such that each cell receives as many channels as its demand requires while respecting the separation constraints. Here, the objective is to minimize the *span* of the assignment which is the difference between the highest and the lowest channel assigned. (An alternative objective, when a limited span is given, can be to minimize the number of violated interference constraints.)

Cellullar networks can be modeled as graphs where the nodes of the graph represent the cells, and two nodes are adjacent precisely when there exists a (non-zero) separation constraint between them. The demands are given by a *weight vector* indexed by the nodes, and the separation constraints are given by a vector indexed by the nodes and edges. When all separation constraints are 1, the CAP reduces to the problem of finding a colouring of a weighted graph.

The minimal span needed for any assignment will generally be determined by the cells with highest demand. It is reasonable to assume that these cells will often be geographically close, corresponding, for example, to a business district or a city center. Since interference also tends to be highest between cells that are close, these cells will often form a clique in the underlying graph.

Most lower bounds for the CAP are therefore based on cliques. The simplest clique bound, mentioned in [5] but generally considered folklore, is found by assuming all edge constraints and co-site constraints are equal to the lowest constraint in the clique. A first refinement was obtained in [5], by considering two different constraints. A second refinement, similar to the situation studied here, was considered in [16]. In all of these cases, bounds were obtained using ad-hoc methods.

In this paper, we study networks where the separation constraint between different cells can take only three values, one of which is reserved for the *co-site constraint*. The co-site constraint is the separation constraint between channels assigned to the same cell, or node. Naturally, any bounds obtained from this approach can also be used in networks with more general constraints by reducing the constraints in any particular set of edges to the lowest constraint in that set.

We describe how lower bounds can be generated from an approach based on reducing the CAP to a covering problem. The crucial step is to show that any channel assignment can be broken down into a small blocks called *tiles*. A *tile cover* is a collection of tiles so that the number of tiles covering a node equals the number of channels assigned to that node. The conversion of the CAP to a tile cover problem brings the advantage that tile covers can be easily analyzed using LP duality and polyhedral methods. A similar tile cover method, applied to the simpler case of cliques with one co-site constraint and one edge constraint, can be found in

[10]. This particular result is used in our paper as the base case for the induction which forms the proof of our main theorem. In [12], heuristic frequency assignment methods using pre-assigned "tiles" of assigned channels are applied successfully to a number of CAP instances.

We apply the tile cover approach to configurations which we call *nested cliques*. These are cliques consisting of an *inner* clique and an *outer* clique where all edge constraints involving an inner clique node take the larger constraint value, while all edge constraints containing only nodes from the outer clique take the smaller value (see Section 2 for a more precise definition). Nested cliques arise naturally from the geographical layout of cellular networks and the fact that interference levels are generally lower between transmitters that are at greater distance from each other. Hence, it will be common to find a cluster of cells with high interference constraints between them surrounded by an outer shell of cells at greater distance, and thus with weaker interference constraints. Such a situation will form a nested clique in the interference graph.

Using the tile cover approach on nested cliques, we derive a comprehensive family of general "second generation" clique bounds. This family includes all bounds from [5], and improves the bound obtained in [16]. We also show, using an example, how the approach can be used directly to obtain specific lower bounds for any specific set of parameters.

There are two types of clique bounds that cannot be derived directly from our approach. In [14] and [7], it was shown how the Traveling Salesman Problem and its Linear Program relaxation can be used to derive lower bounds for cliques. This approach is most effective when the co-site constraint is relatively low. In [17] a lower bounding method is described which is based on network flows. However, our tile cover bounds give better results when applied to the example given in this paper.

Since it is NP-hard to find a maximum weight clique in a graph, it will also be hard to find the nested clique that gives the best bound. However, clique enumeration procedures such as the *Cardaghan-Pardalos* algorithm (see [4]) give good performance in practice. The reduction of the CAP to a tile cover problem leads to an easy way of computing the lower bound for any particular clique by way of a linear program. Alternatively, any particular network can be analyzed in advance using our method, and a complete family of easily computable lower bounds can be obtained. Therefore, we expect the computation of the best tile cover clique bound to be feasible and realistic.

In Section 3, we describe the tiles that can occur in a tile cover for nested cliques. A *cost* is associated with each tile, which roughly corresponds to the part of the span taken up by assigning channels to the tile. Our main result, proven in Section 5, states that each channel assignment can be reduced to a tile cover, such that the cost of the cover is no larger than the span of the assignment. This then implies that any lower bound on the cost of a tile cover is a lower bound on the span of a channel assignment.

In Section 4, we develop lower bounds for tile covers, which then directly translate into bounds for the CAP. First we formulate the Integer Program which finds tile covers of minimal cost, and then use its LP relaxation, LP duality and polyhedral methods to obtain lower bounds. We show how this approach generates or generalizes the bounds from [5] and [16]. Moreover, we show how the same method could be used to generate lower bounds for any particular choice of parameters. We demonstrate this approach on an instance of the CAP taken from [17], where our methods give an improvement of 13% over the previously best bound.

2 Preliminaries

For the basic definitions of graph theory we refer to [3]. A (simple) graph G is a pair (V, E) of a node set V and an edge set E, where E is a set of 2-subsets of V. A **clique** in a graph is a set of nodes of which every pair is adjacent.

In this paper, we will use the following notation for integer vectors: if $y \in \mathbb{Z}^V$ for some set V, then $y(v)$ is the coordinate of y indexed by v. Sets will often be represented by their characteristic vectors. Given a set V and $A \subseteq V$, the characteristic vector $\chi^A \in \mathbb{Z}_+^V$ is defined as follows:

$$\chi^A(v) = \begin{cases} 1 & \text{if } v \in A \\ 0 & \text{otherwise} \end{cases}$$

Conversely, given a vector $y \in \mathbb{Z}_+^V$, the **support** of y, denoted by $V(y)$, is the set of all nodes in V indexing non-zero coordinates of y, so

$$V(y) = \{v \in V : y(v) > 0\}.$$

A **constrained graph** $G = (V, E, s, e)$ is a graph $G = (V, E)$ and positive integer vectors $s \in \mathbb{Z}_+^V$ and $e \in \mathbb{Z}_+^E$ representing the reuse constraints: the vector s represents the **co-site constraints**, the required separation between channels assigned to the same node, and e represents the **edge constraints**, the required separation between channels assigned to the two endpoints of an edge.

A **constrained, weighted graph** is a pair (G, w) where G is a constrained graph and w is a positive integral weight vector indexed by the nodes of G. The coordinate of w corresponding to node u is denoted by $w(u)$ and called the **weight** of node u. The weight of node u represents the number of channels needed at node u.

A **channel assignment** for a constrained, weighted graph (G, w) where $G = (V, E, s, e)$ is an assignment f of sets of non-negative integers (which will represent the channels) to the nodes of G which satisfies the conditions:

$$\begin{array}{ll} |f(u)| = w(u) & (u \in V), \\ i \in f(u) \text{ and } j \in f(v) \Rightarrow |i - j| \geq e(uv) & (uv \in E,\, u \neq v), \\ i, j \in f(u) \text{ and } i \neq j \Rightarrow |i - j| \geq s(u) & (u \in V). \end{array}$$

For reasons of brevity, throughout this paper we will use the notation $f(V)$ to denote $f(V) = \bigcup_{u \in V} f(u)$, in deviation from the standard definition of $f(V) = \{f(u) \,|\, u \in V\}$.

The **span** $S(f)$ of a channel assignment f of a constrained weighted graph is the difference between the lowest and the highest channel assigned by f, in other words, $S(f) = \max f(V) - \min f(V)$. The span $S(G, w)$ of a constrained, weighted graph G and a positive integer vector w indexed by the nodes of G is the minimum span of any channel assignment for (G, w).

We will consider complete graphs with constraints that have a special, nested structure. A constrained graph $G = (V, E, s, e)$ is a **nested clique** with parameters (k, u, a), where $k \geq u \geq a$, if $s(v) \geq k$ for all $v \in V$, and V can be partitioned into two sets Q and R such that $e(vw) \geq a$ if $v, w \in R$, and $e(vw) \geq u$ otherwise. The parameters k, u and a are always assumed to be positive integers. We can also assume that $k > 1$. Otherwise, we would have $k = u = a = 1$, and $S(G, w) = \sum_{v \in V} w(v) - 1$

3 Tile Covers

In this paper, we reduce the channel assignment problem for nested cliques to a tile covering problem. The tiles that may be used for a tile cover are defined in this section. We can think of these tiles as partial assignments, or 'building blocks', from which any possible assignment can be constructed.

We assume that a particular nested clique G with node partition (Q, R) and parameters (k, u, a) is given. We define the set $\mathcal{T}$ of all possible tiles that may be used in a tile cover of G. All tiles are defined as vectors indexed by the nodes of G. For reasons of brevity we will sometimes identify a tile with its support, and thus think of tiles as node sets. It is this representation that allows mention of 'the nodes in tile t'.

In order to facilitate the definition and the proof of Theorem 5.1, we distinguish various categories of tiles. So

$$\mathcal{T} = \mathcal{T}_Q \cup \mathcal{T}_R \cup \mathcal{T}_{QR} \cup \mathcal{T}_{QR}^{big}.$$

The tiles in each category are defined below.

$$\begin{aligned}
\mathcal{T}_Q &= \{\chi^A : A \subseteq Q\},\\
\mathcal{T}_R &= \{\chi^B : B \subseteq R\},\\
\mathcal{T}_{QR} &= \{\chi^A + \chi^B : A \subseteq Q,\ B \subseteq R \text{ where } A \neq \emptyset,\ B \neq \emptyset\}\\
\mathcal{T}_{QR}^{big} &= \{\chi^{A\cup B} + \chi^{A_2 \cup B_2} : A_2 \subseteq A \subseteq Q,\ B_2 \subseteq B \subseteq R,\ A_2 \neq \emptyset,\ B_2 \neq \emptyset\}
\end{aligned}$$

The tiles in $\mathcal{T}_{QR}^{big}$ will be called **big** tiles. Note that all coefficients of tiles in $\mathcal{T}_Q$, $\mathcal{T}_R$ and $\mathcal{T}_{QR}$ have value either zero or one, while for tiles in $\mathcal{T}_{QR}^{big}$, the coefficients indexed by nodes in A_2 and B_2 have value 2.

A **tiling** is a collection of tiles from $\mathcal{T}$ (multiplicities are allowed). We represent a tiling by a non-negative integer vector $y \in \mathbb{Z}_+^{\mathcal{T}}$, where $y(t)$ represents the number of copies of tile t present in the tiling. A **tile cover** of a weighted nested clique (G, w) is a tiling y such that $\sum_{t\in\mathcal{T}} y(t)t(v) \geq w(v)$ for each node v of G.

With each tile $t \in \mathcal{T}$ we associate a cost $c(t)$. The costs of the tiles in each category are given in Table 1. The cost of each tile t is derived from the span of a channel assignment for (G, t) plus a 'link-up' cost of connecting the assignment to a following tile. This 'link-up' cost is calculated using the assumption that the same assignment will be repeated. For example, $t = \chi^A$, where $A = \{v_0, \ldots, v_{j-1}, v_j\}$, is a tile of $j + 1$ distinct vertices in Q. Then the minimum span of (G, t) is u, and an assignment of minimum span would be $f(v_i) = iu$ for all i. However, if this assignment is repeated, the next channel that can be assigned will be $(j + 1)u$, which is u more than the highest channel in the assignment. Hence the "link-up" cost of this assignment equals u.

In other words, the cost of a tile t is such that for any constant α the minimum span of $(G, \alpha t)$ equals $\alpha c(t)$ minus a small constant, or

$$\frac{S(G, \alpha t)}{\alpha} \rightarrow c(t) \text{ as } \alpha \rightarrow \infty.$$

It will follow from Theorem 5.1 that our choice of the costs is justified.

The **cost** of a tiling y, denoted by $c(y)$, is the sum of the cost of the tiles in the tiling. So $c(y) = \sum_{t\in\mathcal{T}} y(t)c(t)$. The minimum cost of a tile cover of a weighted nested clique (G, w) will be denoted by $\tau(G, w)$.

4 Polyhedral Bounds from Tile Covers

In Section 5 we will prove the following theorem.

Theorem 5.1. *Let G be a nested clique with node partition (Q, R) and parameters (k, u, a). Then for any weight vector w for G,*

$$S(G, w) \geq \tau(G, w) - k.$$

Category	Number of nodes in Q	Number of nodes in R	Cost
$\mathcal{T}_Q$	n	0	$\max\{k, nu\}$
$\mathcal{T}_R$	0	m	$\max\{k, ma\}$
$\mathcal{T}_{QR}$	n	m	$\max\{k, nu + ma + u - a\}$
$\mathcal{T}_{QR}^{big}$	n, of which n_2 have value 2	m, of which m_2 have value 2	$\max\{k, nu\} + \max\{k, ma\}$ $+n_2u + m_2a + u - a$

Table 1: Costs of tiles

In this section, we will demonstrate how this theorem, combined with polyhedral methods, leads to new lower bounds for $S(G, w)$.

The problem of finding a minimum cost tile cover of (G, w) can be formulated as an integer program (IP):

$$
\begin{array}{ll}
\text{Minimize } \sum_{t \in \mathcal{T}} c(t)y(t) & \\
\text{subject to:} & \\
\quad \sum_{t \in \mathcal{T}} t(v)y(t) \geq w(v) & (v \in V) \\
\quad y(t) \geq 0 & (t \in \mathcal{T}) \\
\quad y \text{ integer} &
\end{array}
$$

We obtain the linear programming (LP) relaxation of this IP by removing the requirement that y must be integral. Any feasible solution to the resulting linear program is called a **fractional** tile cover. The minimum cost of a fractional tile cover gives a lower bound on the minimum cost of a tile cover. The dual of this LP is formulated as follows.

$$
\begin{array}{ll}
\text{Maximize } \sum_{v \in V} w(v)x(v) & \\
\text{subject to:} & \\
\quad \sum_{v \in V} t(v)x(v) \leq c(t) & (t \in \mathcal{T}) \\
\quad x(v) \geq 0 & (v \in V)
\end{array}
$$

By linear programming duality, the maximum of the dual is equal to the minimum cost of a fractional tile cover. Thus, any vector that satisfies the inequalities of the dual program gives a lower bound on the cost of a minimum fractional tile cover, and therefore also on the span of the corresponding complete constrained, weighted graph. The maximum is achieved by one of the vertices of the polytope $TC(G)$ defined as follows:

$$TC(G) = \{x \in \mathbb{Q}_+^V : \sum_{v \in V} t(v)x(v) \leq c(t) \text{ for all } t \in \mathcal{T}\}.$$

A classification of the vertices of this polytope will therefore lead to a comprehensive set of lower bounds that can be obtained from fractional tile covers. The next theorem demonstrates the strength of the tile cover approach, by giving a family of bounds for nested cliques with parameters $(k, u, 1)$.

Theorem 4.1 *Let G be a nested clique with node partition (Q, R) and parameters $(k, u, 1)$. Let $w \in \mathbb{Z}_+^V$ be a weight vector for G, and let w_{Qmax} be the maximum weight of any node in Q, and w_{Rmax} the maximum weight of any node in R. Then*

$$\tau(G, w) \geq (\lambda_1 - \lambda_2) w_{Qmax} + \lambda_2 \sum_{v \in Q} w(v) + (\lambda_3 - \lambda_4) w_{Rmax} + \lambda_4 \sum_{v \in R} w(v),$$

for each 4-tuple $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_1, λ_2, λ_3 and λ_4 can take the following values:

λ_1	λ_2	λ_3	λ_4	*Case*
k	*0*	*0*	*0*	*(1)*
0	*0*	k	*0*	*(2)*
$k-(\mu-1)\delta$	δ	δ	0	*(3)*
δ	δ	$k-(\mu-1)\delta$	0	*(4)*
$k-(\mu-1)\delta$	δ	ϵ	ϵ	*(5)*
u	u	*1*	*1*	*(6)*
u	u	u	$\frac{k-u}{k-1}$	*(7)*
$2u-1$	ν	*1*	*1*	*(8)*

where $\mu = \lfloor \frac{k}{u} \rfloor$, $\delta = (\mu+1)u - k$, $\epsilon = \begin{cases} 1 & \textit{if } \mu = 1 \\ \min\{\frac{\delta}{k-2u+1}, \frac{2u+\mu\delta-\delta}{k+1}, 1\} & \textit{otherwise} \end{cases}$, *and*
$\nu = \begin{cases} 1 & \textit{if } \mu = 1 \\ u - \max\{\frac{u-1}{\mu}, \frac{\delta-1}{\mu-1}\} & \textit{otherwise} \end{cases}$.

Proof. For the proof we consider feasible points in $TC(G)$ that are of the form $\lambda_1 \chi^{\{q\}} + \lambda_2 \chi^{Q-\{q\}} + \lambda_3 \chi^{\{r\}} + \lambda_4 \chi^{R-\{r\}}$, where $q \in Q$ and $r \in R$, and $\lambda_1 \geq \lambda_2$, $\lambda_3 \geq \lambda_4$.

For such points, the inequality system that defines $TC(G)$ reduces to the following form:

$$
\begin{aligned}
&\lambda_1 + (\mu - 1)\lambda_2 \le k \\
&\lambda_1 + \mu\lambda_2 \le (\mu + 1)u \\
&\lambda_3 + (k - 1)\lambda_4 \le k \\
&\lambda_1 + (n - 1)\lambda_2 + \lambda_3 + (m - 1)\lambda_4 \le \max\{k, nu + m + u - 1\} \text{ for all } m, n > 0, \text{ and} \\
&\hspace{25em} n < \mu \text{ or } m < k. \\
&2\lambda_1 + (\mu - 1)\lambda_2 + 2\lambda_3 + (k - 1)\lambda_4 \le 2k + 2u \\
&2\lambda_1 + \mu\lambda_2 + 2\lambda_3 + (k - 1)\lambda_4 \le k + (\mu + 3)u \\
&\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0
\end{aligned}
$$

The first and second inequalities are obtained by choosing tiles of size μ and $\mu + 1$, respectively, from $\mathcal{T}_Q$. The inequalities corresponding to smaller tiles from $\mathcal{T}_Q$ are implied by this first inequality since $\lambda_2 \ge 0$ and the cost of any such tile is k. The inequalities corresponding to larger tiles from $\mathcal{T}_Q$ are implied by the second inequality since the cost of a tile in $\mathcal{T}_Q$ never increases by more than u if a node from Q is added, and $\lambda_1 \ge \lambda_2$ implies that $\lambda_2 \le u$.

The third inequality is derived from a tile of size k from $\mathcal{T}_R$. The inequalities corresponding to smaller tiles from $\mathcal{T}_R$ are implied since $\lambda_4 \ge 0$ and the cost of any such tile is k. Furthermore, the third inequality also implies that $\lambda_4 \le 1$, and since the cost of a tile in $\mathcal{T}_R$ never increases by more than 1 if a node from R is added, the inequalities that correspond to larger tiles from $\mathcal{T}_R$ are also implied.

The forth inequality is derived from a tile in $\mathcal{T}_{QR}$ where $n < \mu$ or $m < k$. The inequalities derived from other tiles in $\mathcal{T}_{QR}$ are implied by the first three inequalities. (If $n = \mu$ and $m \ge k$, the sum of the first and third inequalities suffice. If $n > \mu$ and $m \ge k$, the second and third inequalities suffice.)

The final two inequalities are obtained by choosing tiles from $\mathcal{T}_{QR}^{big}$ where nodes q and r have weight two, all others have weight one, and $m = k$. We have $n = \mu$ and $n = \mu + 1$ in the fifth and sixth inequalities, respectively. Note that the cost of a tile in $\mathcal{T}_{QR}^{big}$ never increases by more than u if a node from Q is either added or has its weight increased from one to two. Similarly, the cost increases by no more than 1 if a node from R is either added or has its weight increased from one to two. Therefore, the inequalities corresponding to tiles of other sizes in $\mathcal{T}_{QR}^{big}$ are implied.

We now turn our attention to verifying that Cases 1 through 8 listed in the Theorem give feasible points. During this process, note that each point satisfies at least one of the inequalities with equality.

All of the cases can be easily verified for $\mu = 1$, so we will only consider $\mu \ge 2$. Cases (1), (2), (6) and (7) are straightforward, keeping in mind that $\mu u \le k$ and $u \ge 1$. We leave verification of these cases to the reader. A discussion of the remaining cases follows.

Case (3). The first three inequalities, as well as the fifth inequality, can be easily verified using the definition of δ and the fact that $\delta \le u \le k$.

To verify the fourth inequality, note that $k - (\mu - 1)\delta + (n - 1)\delta + \delta = (\mu + 1)u + (n - \mu)\delta$. If $n \geq \mu$, we use the substitution $\delta \leq u$ to obtain $(\mu + 1)u + (n - \mu)\delta \leq nu + u + m - 1$. Otherwise, the substitution $n \leq \mu - 1$ is used to obtain $(\mu + 1)u + (n - \mu)\delta \leq k$. The fourth inequality follows.

For the sixth inequality, note that $2(k - (\mu - 1)\delta) + \mu\delta + 2\delta = k + (\mu + 1)u + 2\delta + (1 - \mu)\delta)$. Since $\mu \geq 1$ and $\delta \leq u$, we have $k + (\mu + 1)u + 2\delta + (1 - \mu)\delta \leq k + (\mu + 3)u$.

Case (4). The first three inequalities are easily verified keeping in mind $\delta \leq u$ and $\mu u \leq k$. The remaining inequalities are identical to those in Case (3).

Case (5). If $\mu = 1$ then the inequalities are easily verified using the fact $k \leq 2u - 1$. Hence, we will assume that $\mu \geq 2$.

The first two inequalities are easily verified, and the third follows from the fact that $\epsilon \leq 1$. To verify the fourth inequality, we need to show that $k + (n - \mu)\delta + m\epsilon \leq \max\{k, nu + u + m - 1\}$ for $n < \mu$ or $m < k$.

If $n \geq \mu$, then $k + (n - \mu)\delta + m\epsilon \leq k + (n - \mu)u + m$. Since $k - \mu u < u$, we have $k + (n - \mu)u + m < nu + u + m$.

Suppose $n < \mu$ and $m \leq k - 2u + 1$. Since $n - \mu \leq -1$ and $\epsilon \leq \frac{\delta}{k - 2u + 1}$, then $k + (n - \mu)\delta + m\epsilon \leq k$. Furthermore, every increase of 1 in m results in an increase of at most $\epsilon \leq 1$ in the cost of the tile. Hence, $k + (n - \mu)\delta + m\epsilon \leq nu + u + m - 1$ for $n < \mu$ and $m > k - 2u + 1$, as well. Hence, the fourth inequality holds for all required values of n and m.

The fifth inequality follows directly from the fact $\epsilon \leq \frac{2u + \mu\delta - \delta}{k + 1}$, while the final inequality uses this fact together with the substitution $k + \delta = (\mu + 1)u$.

Case (8). The first and fifth inequalities follow from the fact that $\nu \leq u - \frac{\delta - 1}{\mu - 1}$ and $(\mu + 1)u - \delta = k$. The second and sixth inequalities follow from the fact that $\nu \leq u - \frac{u - 1}{\mu}$. The forth follows from the fact that $\nu \leq u$. The third inequality is straightforward.

So for each vector $x = \lambda_1\chi^{\{q\}} + \lambda_2\chi^{Q - \{q\}} + \lambda_3\chi^{\{r\}} + \lambda_4\chi^{R - \{r\}}$ with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ as given in the table, and q and r any nodes in Q and R, respectively, it holds that $x \in TC(G)$, and thus $\tau(G, w) \geq \sum_{v \in V} w(v)x(v)$. Since $\lambda_1 \geq \lambda_2$ and $\lambda_3 \geq \lambda_4$, $\sum_{v \in V} w(v)x(v)$ is maximized when we choose q and r to be the nodes of maximum weight in Q and R, respectively. With this choice of q and r, $\sum_{v \in V} w(v)x(v) = (\lambda_1 - \lambda_2)w_{Qmax} + \lambda_2 \sum_{v \in Q} w(v) + (\lambda_3 - \lambda_4)w_{Rmax} + \lambda_4 \sum_{v \in R} w(v)$, and the result follows. □

Theorem 4.1 leads to a family of bounds, since each case of values for the parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ as given in the table leads to a different bound. Some of these bounds are new, while others have been obtained before by conventional methods.

The bounds derived from Cases (5), (7) and (8) are new. From Case (7), where

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (u, u, u, \frac{k-u}{k-1})$ we obtain the bound

$$S(G, w) \geq u(\sum_{v \in Q} w(v) + w_{R\max}) + \frac{k-u}{k-1} \sum_{v \in R, v \neq v_{Rmax}} w(v) - k.$$

This bound strengthens the bound $S(G, w) \geq u \sum_{v \in C} w(v) - u$ (first mentioned in [5]), which holds for any clique C with where all edge constraints have value at least u.

From Case (8), which uses the point $(2u-1, \nu, 1, 1)$ we obtain the new bound

$$S(G, w) \geq (2u-1) w_{Qmax} + \nu \sum_{v \in Q, v \neq v_{Qmax}} w(v) + \sum_{v \in R} w(v) - k.$$

In [15] a bound of $(2u-1)w_{Qmax} + \sum_{v \in R} w(v) - \kappa$ (where κ is a small constant) is given for nested cliques with the special property that $|Q| = 1$. The bound resulting from Case (8) can be seen as an generalization of this bound for nested cliques where Q contains more than one node.

Case (5) uses the point $(k - (\mu - 1)\delta, \delta, \epsilon, \epsilon)$ and leads to the bound

$$S(G, w) \geq (k - \mu\delta) w_{Qmax} + \delta \sum_{v \in Q} w(v) + \epsilon \sum_{v \in R} w(v) - k.$$

The new bound from Case (5) can be seen as an extension of the bound $S(G, w) \geq (k - \mu\delta)w_{max} + \delta \sum_{v \in C} w(v) - \kappa$ (κ is a small constant) that was given for cliques with co-site constraint k and uniform edge constraint u in [5].

Using the clique $Q \cup \{v_{Rmax}\}$ (with edge constraint at least u), our method also gives the bound

$$S(G, w) \geq (k - \mu\delta) w_{max} + \delta(\sum_{v \in Q} w(v) + w_{Rmax}) - k.$$

We simply use Case (3) or Case (4), depending on whether $w_{max} = w_{Qmax}$ or $w_{max} = w_{Rmax}$, respectively.

The bound from Case (6), namely

$$S(G, w) \geq u \sum_{v \in Q} w(v) + \sum_{v \in R} w(v) - k$$

was the first bound treating nested cliques specifically. It was derived in [5] using ad hoc methods.

The bound derived from Cases (1) and (2) is the well known bound

$$S(G, w) \geq k w_{max} - k.$$

In all these results, we have used the general rule, stated in Theorem 5.1 that $S(G,w) \geq \tau(G,w) - k$. A careful reading of the proof of Theorem 5.1 will show that in most cases the extra term k is too pessimistic. In principle, it is possible to find a more precise additive term by a more precise, and hence more complicated, analysis. Since our main interest here lies in showing a *method* by which lower bounds can be derived, rather than finding the best possible lower bounds, we contented ourselves with the additive factor of k. However, this may cause our bounds to differ slightly from the older bounds.

The following theorem gives two new bounds for another variation of the parameters (k,u,a).

Theorem 4.2 *Let G be a nested clique with node partition (Q,R) and integer parameters (k,u,a), where $\lfloor \frac{k}{u} \rfloor = 1$ and $\lfloor \frac{k}{a} \rfloor = 2$. Let $w \in \mathbb{Z}_+^V$ be a weight vector for G, and let w_{Qmax} be the maximum weight of any node in Q, and w_{Rmax} the maximum weight of any node in R. Then*

$$S(G,w) \geq u(\sum_{v \in Q} w(v) + w_{Rmax}) + \alpha \sum_{v \in R, v \neq v_{Rmax}} w(v) - k.$$

where $\alpha == \min\{\frac{3a-u}{2}, k-u\}$ and

$$S(G,w)) \geq \beta \sum_{v \in Q} w(v) + (2k-3a) w_{Rmax} + (3a-k) \sum_{v \in R} w(v) - k,$$

where $\beta = \min\{2u + 3a - 2k, u\}$.

Proof. For the proof we again consider feasible points in $TC(G)$ that are of the form $\lambda_1 \chi^{\{q\}} + \lambda_2 \chi^{Q-\{q\}} + \lambda_3 \chi^{\{r\}} + \lambda_4 \chi^{R-\{r\}}$, where $q \in Q$ and $r \in R$, and $\lambda_1 \geq \lambda_2$, $\lambda_3 \geq \lambda_4$.

For such points, and for parameters as mentioned in the theorem, the inequality system that defines $TC(G)$ reduces to the following form:

$$\begin{aligned}
&\lambda_1 \leq k \\
&\lambda_1 + \lambda_2 \leq 2u \\
&\lambda_3 + \lambda_4 \leq k \\
&\lambda_3 + 2\lambda_4 \leq 3a \\
&\lambda_1 + \lambda_3 \leq 2u \\
&\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0
\end{aligned}$$

It is straightforward to check that (u,u,u,α) where $\alpha = \min\{\frac{3a-u}{2}, k-u\}$ and $(\beta, \beta, 2k-3a, 3a-k)$ where $\beta = \min\{2u + 3a - 2k, u\}$ are feasible points of this system. When we choose the vectors of this form so that the maximum coordinates correspond to the nodes of maximum weight, the bounds follow. □

The preceding theorems show how new lower bounds can be generated for any particular choice of parameters. In practice, it will often be useful to apply the tile cover method directly to the exact parameters of the particular network. For any specific nested clique, a classification of all extreme points of $TC(G)$ can be obtained by using vertex enumeration software, for example the package `lrs`, developed by David Avis [2]. In general, we can use the dual program to obtain families of vertices, and hence bounds, for certain choices of parameters.

This approach is demonstrated in the following example. The example is taken from [17], where it was used to demonstrate a lower bound derived from network flows. We will see that our tile cover approach gives a significant improvement.

Example 4.1 *Consider the cellular network layout as shown Figure 4.1. The circled numbers in each cell represent the label of the cell; the node associated with the cell with label i is called v_i. The larger number in each cell gives the demand in the cell, i.e. the weight of the associated node. The particular hexagonal cell layout of this example is that of the "Philadelphia problem" [1], which has been frequently used as a benchmark for algorithms and lower bounds for the channel assignment problem (see for example [5],[6],[9],[12],[13],[18]).*

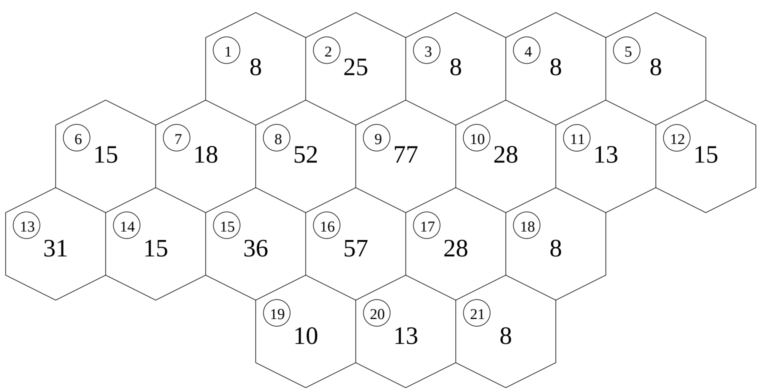

Figure 1: The layout of the example.

The constraints are described in terms of the distance d_{ij} between the centers of cells v_i and v_j where the unit is the distance between the centers of adjacent cells.

$$c_{ij} = \begin{cases} 0 & \text{if } d_{ij} > 3, \\ 1 & \text{if } \sqrt{3} < d_{ij} \leq 3, \\ 2 & \text{if } 0 < d_{ij} \leq \sqrt{3}, \\ 5 & \text{if } i = j \end{cases}$$

This layout contains nested cliques of size 8, with 2 nodes in Q and 6 nodes in R, and nested cliques of size 7, with one node in Q and 6 nodes in R. The nested cliques have parameters $(5, 2, 1)$.

For a nested clique with bipartition (Q, R) where $|Q| = 2$ and $|R| = 6$, we derived a set of lower bounds using the software `lrs`*. We looked for points of the form $(x_1, x_2, y_1, y_2, y_3, y_4, y_5, y_6)$, where x_1 and x_2 correspond to nodes of Q and $x_1 \geq x_2$, and $y_1, \ldots, y_6$ correspond to the nodes of R, and $y_1 \geq y_2 \geq ... \geq y_6$. The inequality system that defines $TC(G)$ reduces to the following:*

$x_1 + x_2 \leq 5$
$y_1 + y_2 + y_3 + y_4 + y_5 \leq 5$
$x_1 + y_1 + y_2 \leq 5$
$x_1 + y_1 + y_2 + y_3 \leq 6$
$x_1 + y_1 + y_2 + y_3 + y_4 \leq 7$
$x_1 + x_2 + y_1 \leq 6$
$x_1 + x_2 + y_1 + y_2 \leq 7$
$x_1 + x_2 + y_1 + y_2 + y_3 \leq 8$
$x_1 + x_2 + y_1 + y_2 + y_3 + y_4 \leq 9$
$x_1 \geq x_2,\ y_1 \geq y_2 \geq \ldots, \geq y_6$
$x_1, x_2, y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$

Given this system, `lrs` *returned a set of vertices, 14 of which could be used to generate lower bounds (the other vertices could be obtained from those 14 by dropping some coordinates to zero).*

We applied these bounds to the nested clique formed by the cells as indicated in Figure 1. Here $Q = \{v_9, v_{16}\}$, and $R = \{v_2, v_8, v_{10}, v_{15}, v_{17}, v_{20}\}$. To obtain best possible results, the nodes of larger weight in Q and R were matched with larger coordinates x_i or y_i, respectively. The best result was obtained by the point $(3, 2, 1, 1, 1, 1, 1, 1)$. The corresponding lower bound is

$$\begin{aligned} S(G, w) &\geq 3w(v_9) + 2w(v_{16}) + \sum_{v \in R} w(v) - 5 \\ &= 3 \cdot 77 + 2 \cdot 57 + (52 + 36 + 28 + 28 + 25 + 13) - 5 \\ &= 522. \end{aligned}$$

This improves by 13% the lower bound of 460 obtained in [17].

5 From Channel Assignments to Tile Covers

In this section we give the proof of the theorem:

Theorem 5.1 *Let G be a nested clique with node partition (Q,R) and parameters (k,u,a). Then for any weight vector w for G,*

$$S(G,w) \geq \tau(G,w) - k.$$

This theorem will follow as a corollary from a more technical lemma. The lemma reduces any channel assignment to a tiling that uses only tiles from $\mathcal{T}$, except for at most one extra tile called a *patch*. A patch is added to take care of the highest channels assigned, for which there is no 'link-up' cost. Patches are defined as follows.

Given a nested clique G with node bipartition (Q,R) and constraints (k,u,a), the patch set $\mathcal{P}$ is defined as follows:

$$\mathcal{P} = \mathcal{P}_Q \cup \mathcal{P}_R \cup \mathcal{P}_{QR} \cup \mathcal{P}_{QR}^{big}.$$

The patches in each category are defined below.

$$\begin{aligned}
\mathcal{P}_Q &= \{\chi^A : A \subseteq Q\} \\
\mathcal{P}_R &= \{\chi^B : B \subseteq R\}, \\
\mathcal{P}_{QR} &= \{\chi^A + \chi^B : A \subseteq Q,\, B \subseteq R,\, A \neq \emptyset,\; B \neq \emptyset\}. \\
\mathcal{P}_{QR}^{big} &= \{\chi^{A\cup B} + \chi^{A_2 \cup B_2} : A_2 \subseteq A \subseteq Q,\, B_2 \subseteq B \subseteq R,\, A_2 \neq \emptyset,\, B_2 \neq \emptyset\}
\end{aligned}$$

The costs of the patches in each category are given in Table 2.

Category	Number of nodes in Q	Number of nodes in R	Cost
$\mathcal{P}_Q$	n	0	$(n-1)u$
$\mathcal{P}_R$	0	m	$(m-1)a$
$\mathcal{P}_{QR}$	n	m	$nu + (m-1)a$
$\mathcal{P}_{QR}^{big}$	n, of which n_2 have weight 2	m, of which m_2 have weight 2	$(n+n_2)u + (m_2-1)a+$ $\max\{k, ma\}$

Table 2: Costs of patches

When we reduce a channel assignment to a tiling, a patch from $\mathcal{P}_R$ will only be used when the first channel is assigned to a node in R, and a patch from either $\mathcal{P}_Q$ or $\mathcal{P}_{QR}^{big}$ will only be used if the first channel is assigned in Q.

For the rest of this section we will adopt the following terminology. Suppose f is a channel assignment for a constrained graph G with node set V, where $f(V) = \{c_0, c_1, \ldots, c_f\}$, with $c_0 \leq c_1 \leq \ldots \leq c_f$. We say that a tiling y of G **covers** channels

c_i to c_j (where $j \geq i$) if y is a tile cover of the subgraph induced by the nodes of G that were assigned channels between c_i and c_j. More precisely, y covers channels $\{c_i, \ldots, c_j\}$ if for each node $v \in V$, $\sum_{t \in \mathcal{T}} y(t)t(v) \geq |f(v) \cap \{c_i, \ldots, c_j\}|$. Also, when y is a tiling and t is a patch or tile, we use $y + \{t\}$ to mean the tiling where one more copy of t is added, so, strictly speaking, the tiling $y + \chi^{\{t\}}$.

We start by stating a lemma that proves that any channel assignment can be reduced to a tile cover for the cliques where there is only one edge constraint, and a co-site constraint. This lemma was proved in [10]. A restatement of the proof can be found in Appendix A.

Lemma 5.2 *[10] Let G be a clique with co-site constraint k and edge constraint u. Let Q be the node set of G, and let the tile set $\mathcal{T}_Q$ and patch set $\mathcal{P}_Q$ be as defined above. Then for any channel assignment of (G, w) of span s there exists a tile cover $y \in \mathbb{Z}^{\mathcal{T}_Q \cup \mathcal{P}_Q}$, which contains exactly one patch, p, of (G, w) with cost at most s. Moreover, the support of p consists of the nodes that receive the last $|V(p)|$ channels of the assignment.*

The proof of Lemma 5.2 provides the following method of constructing the tile cover y, with patch p. Begin by finding the set of nodes that are assigned channels in the range $[c_0, c_0 + k)$. Let V_0 denote that set. For $j \geq 1$, we recursively define V_j to be the nodes assigned channels in the range $[c_{j'}, c_{j'} + k)$ where $c_{j'}$ is the first channel not covered by the tiling $\chi^{V_0} + \cdots + \chi^{V_{j-1}}$. This continues until we have a tiling $y = \chi^{V_0} + \cdots + \chi^{V_l}$ that covers all the channels. The final tile χ^{V_l} is taken to be the patch p.

When y contains at least one tile in addition to $p = V_l$, it is shown that the cost of $\chi^{V_0} + \cdots + \chi^{V_{j-1}}$ is at most $c_{j'} - c_0$, for $1 \leq j \leq l$. Hence, $c(y - \{p\}) \leq c_{f-n} - c_0$ where $n = |V(p)|$. This is a result of particular note, as it is used throughout the remainder of the paper.

We are now ready to state and prove the technical lemma from which Theorem 5.1 will follow. The proof of this lemma uses a straightforward induction on the number of times the channel assignment "crosses over" from Q to R or vice versa. The base case can be directly derived from Lemma 5.2. For the induction step, three different tilings are obtained. By invocation of Lemma 5.2, tilings are obtained for the first parts of the channel assignment up to the first crossover and between the first and second crossover, respectively. Then induction is used to obtain a tiling of the channel assignment that includes all channels after the second crossover. These three tilings are then combined to obtain one new tiling which satisfies the induction hypothesis. The difficulties arise mainly from the fact that three different patches must be combined. Because of the different types of patches, there are a number of cases that must be considered.

In order to demonstrate some of the different cases, we will refer to the following example. The channels are presented in terms of the nodes to which they have been assigned and in terms of their crossovers. Note that when tilings are presented in subsequent examples, each tiling is expressed as a sum of individual tiles together with at most one tiling. That is, $\chi^A + \chi^B$ will always represent the sum of two tiles, as opposed to a single tile from $\mathcal{T}_{QR}^{big}$, for example. Furthermore, no tiling will have more than one patch. In general, the patch will be the last term listing in the tiling It will explicitly stated as to which tile is serving as the patch.

Example 5.1 *Consider the complete graph G with node set $\{q_1, \dots, q_5, r_1, \dots, r_4\}$, and channel assignment as given in the Table 3. The graph G is a nested clique with parameters $k = 5$, $u = 2$ and $a = 1$.*

We will use the constructions that appear throughout the paper, together with the induction hypothesis to construct the tiling

$$y = \chi^{\{q_1,q_4\}} + \chi^{\{q_1,r_1,q_3\}} + \chi^{\{q_3,q_4\}} + \chi^{\{q_2,r_1,q_5\}} + \chi^{\{q_4\}} + \chi^{\{q_4,q_1\}} +$$

$$\chi^{\{q_3,q_1,r_2,r_3,r_4,r_1\}} + \chi^{\{r_2,r_3,r_4\}} + \chi^{\{r_1,r_2,r_3\}} + \chi^{\{q_1,q_2,r_1,r_4\}} + p'$$

where $p' = \chi^{\{r_3,r_4\}} + \chi^{\{q_1,q_5,r_1,r_3\}} + \chi^{\{q_5\}} \in \mathcal{P}_{QR}^{big}$. This tiling covers the graph and its cost is less than the span of the channel assignment.

Vertex	Channels	Vertex	Channels
q_1	0, 15, 22, 50, 56, 63	r_1	12, 24, 39, 61, 72, 78
q_2	17, 37	r_2	58, 67, 73
q_3	26, 31, 54	r_3	7, 13, 59, 69, 74
q_4	2, 34, 43, 48	r_4	8, 60, 70, 79
q_5	5, 41, 65		

Channels assigned to Q	Channels assigned to R
0, 2, 5	7, 8, 12, 13
15, 17, 22	24
26, 31, 34, 37	39
41, 43, 48, 50, 54, 56	58, 59, 60, 61
63, 65	67, 69, 70, 72, 73, 74, 78, 79

Table 3: Channel Assignment

Lemma 5.3 *Let G be a nested clique with node partition (Q, R) and integer constraints (k, u, a), and let $\mathcal{T}$ and $\mathcal{P}$ be the tile and patch set for G. Let f be a channel assignment for G, where $f(V) = \{c_0, c_1, \ldots, c_f\}$, $c_0 < c_1 < \ldots < c_f$. Then there exists a tile cover $y \in \mathbb{Z}_+^{\mathcal{T}\cup\mathcal{P}}$ of (G, w) which contains one patch p, covers all channels $\{c_0, \ldots, c_f\}$, and has cost at most $c_f - c_0$.*

Furthermore, if c_0 is assigned to a node in Q then $p \notin \mathcal{P}_R$, and if c_0 is assigned to a node in R then $p \notin \mathcal{P}_Q \cup \mathcal{P}_{QR}^{big}$.

Proof. Let G be a nested clique as defined in the statement of the lemma. We will prove the lemma by induction on the number of *crossovers* of the channel assignment. A crossover is a pair of channels (c_i, c_{i+1}) where the nodes that receive channels c_i and c_{i+1} are in different part of the bipartition (Q, R).

Base Case

If f is a channel assignment for G with no crossovers, then the statement follows directly from Lemma 5.2.

Let f be a channel assignment with one crossover, and $f(V) = \{c_0, c_1, \ldots, c_f\}$, where $c_0 < c_1 < \ldots < c_f$.

Suppose that c_0 is assigned to a node in Q. Let c_ℓ be the first channel in R greater than c_0. By Lemma 5.2, we can cover the channels in $\{c_0, \cdots, c_{\ell-1}\}$ (which are all assigned to nodes in Q) with a tiling y_Q, containing one patch $p_Q \in \mathcal{P}_Q$, with cost at most $c_{\ell-1} - c_0$. Likewise, the channels in $\{c_\ell, \cdots, c_f\}$ can be covered with a tiling y_R of cost at most $c_f - c_\ell$ containing one patch $p_R \in \mathcal{P}_R$. Combining the two patches into one, we form a new patch $p' = p_Q + p_R \in \mathcal{P}_{QR}$ with cost $nu + (m-1)a$, where $n = |V(p_Q)|$ and $m = |V(p_R)|$. So $c(p') = c(p_Q) + c(p_R) + u$. Moreover, $c_\ell - c_{\ell-1} \geq u$ since $c_{\ell-1}$ is assigned to a nodes in Q, and c_ℓ to a node in R.

Our final tiling is $y = y_Q - \{p_Q\} + y_R - \{p_R\} + \{p'\}$ with cost

$$\begin{aligned} c(y) &= c(y_Q) + c(y_R) + (c(p') - c(p_Q) - c(p_R)) \\ &\leq (c_{\ell-1} - c_0) + (c_f - c_\ell) + u \\ &= c_f - c_0 - (c_\ell - c_{\ell-1} - u) \\ &\leq c_f - c_0. \end{aligned}$$

When c_0 is assigned to a node in R, the proof is analogous.

Example 5.2 *We will find a tiling of G in Example 5.1, restricted to the channels 63 through 79. There is one crossover, (65, 67), in this restricted assignment. The tiling y_Q must cover channels in $\{63, 65\}$, while y_R covers those in $\{67, 69, 70, 72, 73, 74, 78, 79\}$. For y_Q, only one tile is required to cover $\{63, 65\}$. Therefore, $y_Q = \{p_Q\} = \chi^{\{q_1,q_5\}}$. For y_R, we form three tiles covering $\{67, 69, 70\}$,*

$\{72,73,74\}$ and $\{78,79\}$, respectively. Hence, $y_R = \chi^{\{r_2,r_3,r_4\}} + \chi^{\{r_1,r_2,r_3\}} + \chi^{\{r_1,r_4\}}$, where the final tile listed is p_R.

Our new tile, y_1, will have patch $p_Q + p_R = \chi^{\{q_1,q_5,r_1,r_4\}}$. Hence, $y_1 = \chi^{\{r_2,r_3,r_4\}} + \chi^{\{r_1,r_2,r_3\}} + \chi^{\{q_1,q_5,r_1,r_4\}}$. The cost of y_1 is $c(y_1) = \max\{5, 3 \cdot 1\} + \max\{5, 3 \cdot 1\} + (2 \cdot 2 + (2-1) \cdot 1) = 15$. Hence, $c(y_1) \le 79 - 63 = 16$.

Induction Step

For the induction step, let f be a channel assignment with g crossovers, where $g \ge 2$, and assume that the lemma holds for any channel assignment with less than g crossovers. Let $f(V) = \{c_0, c_1, \ldots, c_f\}$, where $c_0 < c_1 < \ldots < c_f$.

CASE 1: Channel c_0 is assigned to a node in Q.
Let c_ℓ be the first channel assigned to a node in R, and c_j the first channel greater than c_ℓ assigned to a node in Q. So $(c_{\ell-1}, c_\ell)$ and (c_{j-1}, c_j) are the first two crossovers of f. Note that $c_\ell \ge c_{\ell-1} + u$ and $c_j \ge c_{j-1} + u$.

By lemma 5.2, we can find a tiling y_Q (with one patch, $p_Q \in \mathcal{P}_Q$) which covers channels $\{c_0, \cdots, c_{\ell-1}\}$ in Q and has cost at most $c_{\ell-1} - c_0$, and a tiling y_R (with one patch, $p_R \in \mathcal{P}_R$) which covers channels $\{c_\ell, \ldots c_{j-1}\}$ and has cost at most $c_{j-1} - c_\ell$.

Define n and m to be the number of nodes in $V(p_Q)$ and $V(p_R)$, respectively. Note that $V(p_Q)$ consists of the nodes that receive channels $\{c_{\ell-n}, \ldots, c_{\ell-1}\}$ and $c(p_Q) = (n-1)u$. Similarly, $V(p_R)$ consists of the nodes that receive channels $\{c_{j-m}, \ldots, c_{j-1}\}$, and $c(p_R) = (m-1)a$.

Case 1A. Tiling y_R contains only the patch p_R.

In this case, patch p_R covers all channels from c_ℓ to c_{j-1}.

(i) If $c_j - c_{\ell-n} \ge k$, then a complete tile cover is obtained as follows:

Step 1 Form a tile $t' = p_Q + p_R$ ($t' \in \mathcal{T}_{QR}$),

Step 2 Find a tiling y_{end} which covers channels $\{c_j, \ldots, c_f\}$ and has cost at most $c_f - c_j$,

Step 3 Form tile cover $y = y_Q - \{p_Q\} + \{t'\} + y_{end}$.

Note that the existence of a tiling y_{end} with the desired properties in Step 2 follows directly from the induction hypothesis.

Since t' contains the same nodes as p_Q and p_R, and hence covers the same channels, it is clear that y covers all channels from c_0 to c_f and contains one patch (the patch from y_{end}). Moreover, since the channel c_j is assigned to a node of Q, the patch of y_{end} is not from $\mathcal{P}_R$. Hence, y has a patch of the required type. It now remains to be proven that $c(y) \le c_f - c_0$.

Since the channels from $c_{\ell-n}$ to c_j cover $n+1$ nodes in Q and m nodes in R, with two crossovers, we have $c_j \geq c_{\ell-n} + (n-1)u + (m-1)a + 2u$. Also, by assumption, $c_j - c_{\ell-n} \geq k$. Therefore, $c_j - c_{\ell-n} \geq \max\{nu + ma + u - a, k\} = c(t')$, and

$$\begin{aligned} c(y) &= c(y_Q - \{p_Q\}) + c(t') + c(y_{end}) \\ &\leq (c_{\ell-n} - c_0) + (c_j - c_{\ell-n}) + (c_f - c_j) \\ &= c_f - c_0 \end{aligned}$$

Example 5.3 *We will find a tiling of G in Example 5.1, restricted to the channels 43 through 79. This channel assignment has three crossovers, $(56, 58)$, $(61, 63)$ and $(65, 67)$. The tiling y_Q will cover channels in $\{43, 48, 50, 54, 56\}$. This requires three tiles, covering $\{43\}$, $\{48, 50\}$ and $\{54, 56\}$, respectively. The tiling y_R will cover channels in $\{58, 59, 60, 61\}$. This only requires a single tile. Hence, $y_Q = \chi^{\{q_4\}} + \chi^{\{q_4,q_1\}} + \chi^{\{q_3,q_1\}}$ and $y_R = \chi^{\{r_2,r_3,r_4,r_1\}}$.*

According to the above notation, $c_j = 63$ and $c_{\ell-n} = 54$. Since channel 43 is assigned to a node in Q, y_R contains only a patch and $c_j - c_{\ell-n} = 63 - 54 \geq 5 = k$, this example falls under Case 1A(i).

We will now form the new tiling, y_2. By Step 1, $t' = p_Q + p_R = \chi^{\{q_3,q_1,r_2,r_3,r_4,r_1\}}$. Step 2 requires a tiling that covers channels 63 through 79. The tiling y_1 formed in Example 5.2 can be used. Finally, by Step 3, $y_2 = \chi^{\{q_4\}} + \chi^{\{q_4,q_1\}} + \chi^{\{q_3,q_1,r_2,r_3,r_4,r_1\}} + y_1$.

Then $c(y_2) = \max\{5, 1 \cdot 2\} + \max\{5, 2 \cdot 2\} + \max\{5, 2 \cdot 2 + 4 \cdot 1 + 2 - 1\} + c(y_1) = 5 + 5 + 9 + 15 = 34$. Hence, $c(y_2) \leq 79 - 43 = 36$. Furthermore, the patch of y_2 is $\chi^{\{q_1,q_5,r_1,r_4\}} \in \mathcal{P}_{QR}$ (the same one used for y_1). So y_2 has the required cost, and the required type of patch.

(ii) Suppose $c_j - c_{\ell-n} < k$, and suppose that there exists a channel c_i in the range $[c_{\ell-n} + k, c_{\ell-n} + k + u)$ which been assigned to a node in Q. (Note that the choice for c_i is unique, since the given range has length less than u.) In this case, the final tile cover is formed as follows.

Step 1 Let A be the set of all nodes that receive channels from $\{c_{\ell-n}, \ldots, c_{i-1}\}$,

Step 2 Form tile $t' = \chi^A \in \mathcal{T}_{QR}$.

Step 3 Find a tiling y_{end} which covers channels $\{c_i, \ldots, c_f\}$ and has cost at most $c_f - c_i$.

Step 4 Form tile cover $y = y_Q - \{p_Q\} + \{t'\} + y_{end}$.

As in Case $1A(i)$, the only statement requiring a non-trivial proof is that $c(y) \leq c_f - c_0$. We proceed with this proof.

Note that no two channels from $\{c_{\ell-n}, \ldots, c_{i-1}\}$ can be assigned to the same node. This is because the co-site constraint on any node is at least k, and by assumption $c_{i-1} < c_{\ell-n} + k$.

Let $n_1 = |A \cap Q|$ and $m_1 = |A \cap R|$. The channels from $\{c_{\ell-n}, \cdots, c_i\}$ are covered by $n_1 + 1$ nodes from Q, m_1 nodes from R, and contain at least two crossovers since both $c_{\ell-1}$ and c_j fall in this range. Therefore, $c_i \geq c_{\ell-n} + (n_1 - 1)u + (m_1 - 1)a + 2u$. Also, $c_i \geq c_{\ell-n} + k$, by definition. Therefore, $c_i - c_{\ell-n} \geq \max\{k, n_1 u + m_1 a + u - a\} = c(t')$, and

$$\begin{aligned} c(y) &= c(y_Q - \{p_Q\}) + c(t') + c(y_{end}) \\ &\leq (c_{\ell-n} - c_0) + (c_i - c_{\ell-n}) + (c_f - c_i) \\ &= c_f - c_0 \end{aligned}$$

Example 5.4 *We will find a tiling of G in Example 5.1, restricted to the channels 31 to 79. This channel assignment has five crossovers. The tiling y_Q will cover channels in $\{31, 34, 37\}$, while y_R covers 39. Therefore, $y_Q = \chi^{\{q_3,q_4\}} + \chi^{\{q_2\}}$ and $y_R = \chi^{\{r_1\}}$.*

By the above notation, $c_j = 41$ and $c_{\ell-n} = 37$. Note that channel 31 is assigned to a node in Q, y_R has only a patch, $c_j - c_{\ell-n} = 41 - 37 = 4 < 5 = k$, and a channel in the range $[37 + 5, 37 + 5 + 3)$, namely 43, has been assigned to a node in Q. Therefore, this example falls into Case 1A(ii).

We let $c_i = 43$ and, by Step 1, let A be the set of nodes receiving channels 37, 39, and 41. Hence, $t' = \chi^A = \chi^{\{q_2,r_1,q_5\}}$. By Step 2, we require a tiling that covers channels 43 through 79. The tiling y_2 from Example 5.3 will suffice.

By Step 4, we obtain the new tiling $y_3 = \chi^{\{q_3,q_4\}} + \chi^{\{q_2,r_1,q_5\}} + y_2$. The patch, p, of y_2 is used as the patch of y_3. From Example 5.3, we know that $p \in \mathcal{P}_{QR}$. Hence, y_3 has a patch of the required type. Furthermore, $c(y_3) = \max\{5, 2 \cdot 2\} + \max\{5, 2 \cdot 2 + 1 \cdot 1 + 2 - 1\} + c(y_2) = 5 + 6 + 34 = 45$. Hence, $c(y_3) \leq 79 - 31 = 48$, as required.

(iii) Suppose $c_j - c_{\ell-n} < k$, and no channel in the range $[c_{\ell-n} + k, c_{\ell-n} + k + u)$ has been assigned to a node in Q. If there is a channel greater than or equal to $c_{\ell-n} + k + u$, let c_i be the first such channel. If no such c_i exists, let $c_{i-1} = c_f$. The final tile cover is formed as follows.

Step 1 Let A be the set of all nodes that receive channels from $\{c_{\ell-n}, \ldots, c_{i-1}\}$.

Step 2 Find a tiling y_{end} which covers channels $\{c_i, \ldots, c_f\}$ and has cost at most $c_f - c_i$. Let p be the patch of y_{end}. (In the case that $c_{i-1} = c_f$ both y_{end} and p are empty.)

Step 3 If $p \in \mathcal{P}_Q \cup \mathcal{P}_{QR} \cup \mathcal{P}_{QR}^{big}$, form tile $t' = \chi^A \in \mathcal{T}_{QR}$, and let $y = y_Q - \{p_Q\} + \{t'\} + y_{end}$.

Step 4 If $p \in \mathcal{P}_R$, then

4a Pick a node $v \in A \cap Q$,

4b Form patch $p' = p + \chi^{\{v\}} \in \mathcal{P}_{QR}$,

4c Form tile $t' = \chi^A - \chi^{\{v\}}$,

4d Form tile cover $y = y_Q - \{p_Q\} + \{t'\} + y_{end} - \{p\} + \{p'\}$.

Step 5 If p is empty, then

5a Form the patch $p' = \chi^A \in \mathcal{P}_{QR}$.

5b Form tile cover $y = y_Q - \{p_Q\} + \{p'\}$.

As before, it is easy to see that y covers all channels from c_0 to c_f. Steps 3, 4 and 5 guarantee that the patch of y is not in $\mathcal{P}_R$, as required. We prove that in all cases, $c(y) \leq c_f - c_0$.

In all cases, let $n_1 = |Q \cap A|$, and $m_1 = |R \cap A|$. Also note that of the channels in $\{c_{l-n}, \ldots, c_{i-1}\}$, only those in the range $[c_{l-n}, c_{l-n} + k)$ are assigned to nodes in Q, and only those in the range $[c_{l-n} + u, c_{l-n} + u + k)$ are assigned to nodes in R. Therefore, no two channels from $\{c_{l-n}, \ldots, c_{i-1}\}$ are assigned to the same node. Furthermore, A contains nodes from both Q and R since both $c_{\ell-n}$ and c_j are covered by A.

In Steps 3 and 4, we assume that there is channel c_i in the required range. Hence, $c_i - c_{\ell-n} \geq k + u$. We now show that $c_i - c_{\ell-n} \geq n_1 u + m_1 a + u - a$

First, suppose c_i is assigned to a node in Q. Since $\{c_{\ell-n}, \ldots, c_i\}$ is covered by $n_1 + 1$ nodes in Q, m_1 nodes in R and contains at least two crossovers, we have $c_i - c_{\ell-n} \geq (n_1 - 1)u + (m_1 - 1)a + 2u$. Therefore, $c_i - c_{\ell-n} \geq n_1 u + m_1 a + u - a$.

If c_i assigned to a node in R then $\{c_{\ell-n}, \ldots, c_i\}$ covers n_1 nodes of Q, $m_1 + 1$ nodes in R and contains at least three crossovers. Hence, $c_i - c_{\ell-n} \geq (n_1 - 2)u + (m_1 - 1)a + 3u = n_1 u + m_1 a + u - a$.

Hence, in Steps 3 and 4, $c_i - c_{\ell-n} \geq \max\{k + u, n_1 u + m_1 a + u - a\}$.

In Step 3, we have $t' = \chi^A \in \mathcal{T}_{QR}$, and $c(t') = \max\{k, n_1 u + m_1 a + u - a\} \leq c_i - c_{\ell-n}$. Therefore,

$$
\begin{aligned}
c(y) &= c(y_Q - \{p_Q\}) + c(t') + c(y_{end}) \\
&\leq (c_{\ell-n} - c_0) + (c_i - c_{\ell-n}) + (c_f - c_i) \\
&= c_f - c_0.
\end{aligned}
$$

In Step 4, a new patch $p' = p + \chi^{\{v\}} \in \mathcal{P}_{QR}$, is formed since p is not of the required type. The cost of this new patch is $c(p') = c(p) + u$. In finding the cost of t' there are two possibilities to consider. If $n_1 > 1$, then $t' = \chi^A - \chi^{\{v\}} \in \mathcal{T}_{QR}$ and $c(t') = \max\{k, (n_1 - 1)u + m_1 a + u - a\}$. If $n_1 = 1$, then $t' \in \mathcal{T}_R$ and

$c(t') = \max\{k, m_1 a\} \le \max\{k, (n_1-1)u + m_1 a + u - a\}$. Now, since $c_i - c_{\ell-n} \ge \max\{k+u, n_1 u + m_1 a + u - a\}$, it follows that $c(t') \le c_i - c_{\ell-n} - u$. Hence the tiling y has cost

$$\begin{aligned} c(y) &= c(y_Q - \{p_Q\}) + c(y_{end}) + (c(p') - c(p)) + c(t') \\ &\le (c_{\ell-n} - c_0) + (c_f - c_i) + u + (c_i - c_{\ell-n} - u) \\ &= c_f - c_0. \end{aligned}$$

In Step 5, we have $c_{i-1} = c_f$. Since $p' \in P_{QR}$, we have $c(p') = n_1 u + (m_1 - 1)a$. Furthermore, since $\{c_{\ell-n}, \ldots, c_f\}$ contains n_1 nodes from Q, m_1 nodes from R and at least two crossovers, $c_f - c_{\ell-n} \ge n_1 u + (m_1-1)a = c(p')$. Therefore,

$$\begin{aligned} c(y) &= c(y_Q - \{p_Q\}) + c(p') \\ &\le (c_{\ell-n} - c_0) + (c_f - c_{\ell-n}) \\ &= c_f - c_0. \end{aligned}$$

Example 5.5 *We will find a tiling of G in Example 5.1, restricted to the channels 22 through 79. This channel assignment has seven crossovers. The tiling y_Q will cover channel 22, while y_R covers channel 24. Hence, $y_Q = \chi^{\{q_1\}}$ and $y_R = \chi^{\{r_1\}}$.*

Since 22 is assigned to a node in Q, y_R contains only a patch, $c_j - c_{\ell-n} = 26 - 22 < 5 = k$, and there is no channel in the range $[22+5, 22+5+2) = [27, 29)$ assigned to a node in Q, this example falls into Case 1A(iii). We let $c_i = 31$ and, by Step 1, let A be the set of nodes receiving channels in $\{22, 24, 26\}$. Step 2 requires a tiling covering channels 31 through 79. We can choose y_3 from Example 5.4 to serve as y_{end}. The patch of y_3 is in $\mathcal{P}_{QR}$, so we proceed to Step 3. Then $t' = \chi^A = \chi^{\{q_1, r_1, q_3\}}$ and $y_4 = \chi^{\{q_1, r_1, q_3\}} + y_3$.

The patch of y_3 serves as the patch for y_4, so y_4 has a patch in $\mathcal{P}_{QR}$. Furthermore, $c(y_4) = \max\{5, 2\cdot 2 + 1\cdot 1 + 2 - 1\} + c(y_3) = 6 + 45 = 51$. Hence, $c(y_4) \le 79 - 22$, as required.

Example 5.6 *We will find a tiling of G in Example 5.1, restricted to the channels 15 through 26. This channel assignment has two crossovers. As in Example 5.5, $y_Q = \chi^{\{q_1, q_2\}} + \chi^{\{q_1\}}$ and $y_R = \chi^{\{r_1\}}$.*

Since 15 is assigned to a node in Q, y_R contains a patch only, $c_j - c_{\ell-n} = 26 - 22 < 5 = k$, and there is no channel in the range $[22+5, 22+5+2) = [27, 29)$ assigned to a node in Q, this example falls into Case 1A(iii). However, in this example $c_f = 26$, so there is no c_i to choose. We let $c_{i-1} = 26$ and proceed to Step 1. Then A is the set of nodes receiving channels in $\{22, 24, 26\}$. By Step 2, *both y_{end} and p are empty, so we proceed to Step 5. By Step 5, $p' = \chi^{\{q_1,r_1,q_3\}}$ and $y = \chi^{\{q_1,q_2\}} + \chi^{\{q_1,r_1,q_3\}}$.*

We see that p' is in $\mathcal{P}_{QR}$ and $c(y) = \max\{5, 2\cdot 2\} + 2\cdot 2 + (1-1)\cdot 1 = 9 \leq 26-15$, as required.

Case 1B. y_R contains a tile other than p_R.

By Lemma 5.2, patch p_R covers channels $\{c_{j-m}, \dots, c_{j-1}\}$, and these channels are all assigned to nodes in R, so $j - m \geq \ell$. Since $(c_{\ell-1}, c_\ell)$ is a crossover, the assignment of channels $\{c_{j-m}, \dots, c_f\}$ has $g-1$ crossovers. Then, by induction, there exists a tiling y_{end} that covers all channels in $\{c_{j-m}, \dots, c_f\}$, contains a patch $p \in \mathcal{P}_{QR} \cup \mathcal{P}_R$, and has cost at most $c_f - c_{j-m}$.

Let $V_Q = V(p) \cap Q$, $V_R = V(p) \cap R$, and let n^p and m^p denote $|V_Q|$ and $|V_R|$, respectively. Note that $c(p) = n^p u + (m^p - 1)a$ if $p \in \mathcal{P}_{QR}$, and $c(p) = (m^p - 1)a$ if $p \in \mathcal{P}_R$.

Choose t to be any tile from y_R other than p_R. Let $V_t = V(t)$ and $m^t = |V_t|$. Note that $t \in \mathcal{T}_R$ and $c(t) = \max\{k, m^t a\}$. Let $V_{p_Q} = V(p_Q)$. Recall that $|V_{p_Q}| = n$ and $c(p_Q) = (n-1)u$.

In Table 4, we show how to combine p_Q, p and t into a new tile t' and a new patch p'.

Case	Condition	Tile t'	Patch p'
(1)	$p \in \mathcal{P}_{QR}$		
(1.1)	(1) and $V_Q \cap V_{p_Q} = \emptyset$	t	$p + p_Q$
(1.2)	(1) and $V_Q \cap V_{p_Q} \neq \emptyset$		
(1.2.1)	(1.2) and $V_R \cap V_t = \emptyset$	$p + t$	p_Q
(1.2.2)	(1.2) and $V_R \cap V_t \neq \emptyset$	there is no t'	$t + p + p_Q$
(2)	$p \in \mathcal{P}_R$	t	$p + p_Q$

Case	Cost $c(t')$	$t' \in$	Cost $c(p')$	$p' \in$
(1.1)	$c(t)$	$\mathcal{T}_R$	$(n + n^p)u + (m^p - 1)a$	$\mathcal{P}_{QR}$
(1.2.1)	$\max\{k, n^p u + m^p a + m^t a + u - a\}$	$\mathcal{T}_{QR}$	$c(p_Q)$	$\mathcal{P}_Q$
(1.2.2)	–	–	$(n + n^p)u + \lvert V_R \cap V_t\rvert a -$ $a + \max\{k, \lvert V_R \cup V_t\rvert a\}$	$\mathcal{P}_{QR}^{big}$
(2)	$c(t)$	$\mathcal{T}_R$	$nu + (m^p - 1)a$	$\mathcal{P}_{QR}$

Table 4: Combining patches

In Cases (1.1), (1.2.1) and (2), we form the new tiling

$$y = y_Q - \{p_Q\} + y_R - \{p_R\} - \{t\} + y_{end} - \{p\} + \{t'\} + \{p'\}.$$

In each of these three cases, $c(t') + c(p') - c(p_Q) - c(p) - c(t) \leq u$. This can be easily verified for Cases (1.1) and (1.2.1). Hence, we will only work through Case (1.2.1).

Note that in Case (1.2.1), the supports of p and t are disjoint. Hence, $c(t') = \max\{k, n^p u + (m^p + m^t)a + u - a\}$. If $k \geq n^p u + (m^p + m^t)a + u - a$, then $k \geq m^t a$ and $c(t') = c(t) = k$. Otherwise $c(t') = n^p u + m^p a + m^t a + u - a$ and $c(t') - c(t) \leq (n^p u + m^p a + m^t a + u - a) - m^t a = c(p) + u$. Hence, $c(t') - c(t) - c(p) \leq u$. Since $c(p') = c(p_Q)$, it follows that $c(t') + c(p') - c(p_Q) - c(p) - c(t) \leq u$.

Hence, for all three cases,

$$\begin{aligned}
c(y) &= c(y_Q) + c(y_R - \{p_R\}) + c(y_{end}) + (c(t') + c(p') - c(p_Q) - c(p) - c(t)) \\
&\leq (c_{\ell-1} - c_0) + (c_{j-m} - c_\ell) + (c_f - c_{j-m}) + u \\
&= c_f - c_0 - (c_\ell - c_{\ell-1} - u) \\
&\leq c_f - c_0.
\end{aligned}$$

In Case (1.2.2), there is no t', so we take the tiling

$$y = y_Q - \{p_Q\} + y_R - \{p_R\} - \{t\} + y_{end} - \{p\} + \{p'\}.$$

Note that, in this case, the supports of t and p are not disjoint, nor are the supports of p and p_Q. Hence, $p' \in \mathcal{P}_{QR}^{big}$. We now show that $c(p') - c(t) - c(p) \leq nu$.

If $|V_R \cup V_t|a \geq k$, then $c(p') = (n + n^p)u + |V_R \cap V_t|a - a + |V_R \cup V_t|a = (n + n^p)u + m^t a + m^p a - a$. Since $c(t) \geq m^t a$, we have $c(p') - c(t) \leq (n + n^p)u + m^p a - a = nu + c(p)$. If $|V_R \cup V_t|a < k$, then it is also the case that $m^t a < k$. Hence, $c(p') = (n+n^p)u + |V_R \cap V_t|a - a + k \leq (n+n^p)u + m^p a - a + k$ and $c(t) = k$. Therefore, $c(p') - c(t) \leq (n + n^p)u + m^p a - a = nu + c(p)$. Hence, $c(p') - c(t) - c(p) \leq nu$ and

$$\begin{aligned}
c(y) &= c(y_Q) - c(p_Q) + c(y_R - \{p_R\}) + c(y_{end}) + (c(p') - c(t) - c(p)) \\
&\leq (c_{\ell-1} - c_0) - (n-1)u + (c_{j-m} - c_\ell) + (c_f - c_{j-m}) + nu \\
&= c_f - c_0 - (c_\ell - c_{\ell-1} - u) \\
&\leq c_f - c_0.
\end{aligned}$$

Hence, in all cases, y covers all channels, the patch of y is of the required type, and $c(y) \leq c_f - c_0$.

Example 5.7 *We will find a tiling of G in Example 5.1. In this example, we are covering all channels 0 through 79. The tiling y_Q will cover channels in $\{0, 2, 5\}$, while y_R covers those in $\{7, 8, 12, 13\}$. Then $y_Q = \chi^{\{q_1,q_4\}} + \chi^{\{q_5\}}$ and $y_R = \chi^{\{r_3,r_4\}} + \chi^{\{r_1,r_3\}}$. Since 0 is assigned to a node in Q and y_R contains a tile other than p_R, this example falls under Case 1B. Then we need a tiling y_{end} that covers channels 12 through 79. It will be shown later in Example 5.8 that there such a tiling, y_5. This tiling has patch $p = \chi^{\{q_1,q_5,r_1,r_3\}} \in \mathcal{P}_{QR}$. Note that $c(p) = 2 \cdot 2 + (2-1) \cdot 1 = 5$.*

We now have $V_Q = \{q_1, q_5\}$, $V_R = \{r_1, r_3\}$ and $V_{P_Q} = \{q_5\}$. The tiling y_R has only one tile besides its patch. Therefore, $V_t = \{r_3, r_4\}$. Since $V_Q \cap V_{P_Q}$ and $V_R \cap V_t$ are both nonempty, this example falls under Case (1.2.2). Hence, there is no tile t', and $p' = \chi^{\{r_3,r_4\}} + \chi^{\{q_1,q_5,r_1,r_3\}} + \chi^{\{q_5\}}$. Hence, $p' \in \mathcal{P}^{big}_{QR}$, and $c(p') = (2+1) \cdot 2 + (1-1) \cdot 1 + \max\{5, 3 \cdot 1\} = 11$.

We now form the final tiling, $y_6 = \chi^{\{q_1,q_4\}} + y_5 - \{p\} + p'$. In Example 5.8, it will be shown that $c(y_5) = 58$. Therefore, $c(y_6) = \max\{5, 2 \cdot 2\} + 58 - 5 + 11 = 69$. Hence, the cost of the tiling for G is less than the span of the channel assignment.

CASE 2: Channel c_0 is assigned to a node in R.

Let c_ℓ be the first channel assigned to a node in Q, and c_j the first channel greater than c_ℓ assigned to a node in R. So $(c_{\ell-1}, c_\ell)$ and (c_{j-1}, c_j) are the first two crossovers of f. Note that $c_\ell \geq c_{\ell-1} + u$ and $c_j \geq c_{j-1} + u$.

By Lemma 5.2, we can find a tiling y_R (with one patch, $p_R \in \mathcal{P}_R$) which covers channels $\{c_0, \cdots, c_{\ell-1}\}$ in R and has cost at most $c_{\ell-1} - c_0$, and a tiling y_Q (with one patch, $p_Q \in \mathcal{P}_Q$) which covers channels $\{c_\ell, \ldots c_{j-1}\}$ and has cost at most $c_{j-1} - c_\ell$.

Let $|V(p_Q)| = n$ and $|V(p_R)| = m$. By Lemma 5.2, $V(p_R)$ consists of the nodes that receive channels $\{c_{\ell-m}, \ldots, c_{\ell-1}\}$ and $c(p_R) = (m-1)a$. Also note that $y_R - \{p_R\}$ covers channels $\{c_0, \ldots, c_{\ell-m-1}\}$ and has cost at most $c_{\ell-m} - c_0$.

Case 2A. Tiling y_Q contains only the patch p_Q.

Since this case is similar to Case 1A, we omit some of the details of the proof.

If $c_j - c_{\ell-m} \geq k$, then we form the tile $t' = p_Q + p_R \in \mathcal{T}_{QR}$. Since there are two crossovers in $\{c_{\ell-m}, \ldots, c_j\}$, we have $c_j \geq c_{\ell-m} + (n-1)u + (m-1)a + 2u$, and thus $c_j - c_{\ell-m} \geq c(t')$. By induction, there exists a tiling y_{end} which covers channels $\{c_j, \ldots c_f\}$, has cost at most $c_f - c_j$, and its patch is not from $\mathcal{P}_Q$ or $\mathcal{P}^{big}_{QR}$.

Let $y = y_R - \{p_R\} + \{t'\} + y_{end}$. Then y covers all channels, and

$$c(y) = c(y_R - \{p_R\}) + c(t') + c(y_{end}) \leq c_f - c_0.$$

Suppose that $c_j - c_{\ell-m} < k$. If there exists a channel in the range $[c_{\ell-m} + k, c_{\ell-m} + k + u)$ which has been assigned to a node in R, then let c_i be the first such channel. If no channel from the range $[c_{\ell-m} + k, c_{\ell-m} + k + u)$ is assigned to a node in R, then let c_i be the first channel greater than or equal to $c_{\ell-m} + k + u$. In either case, let w be the node that has been assigned c_i, let y_{end} be a tiling covering channels $\{c_i, \ldots, c_f\}$ of cost at most $c_f - c_i$, and let p be the patch of y_{end}. If no channel c_i can be selected, then let $c_{i-1} = c_f$. In all cases, A denotes the set of nodes receiving channels from $\{c_{\ell-m}, \ldots, c_{i-1}\}$. Let $n_1 = |A \cap Q|$ and $m_1 = |A \cap R|$.

If $p \in \mathcal{P}_R \cup \mathcal{P}_{QR}$, then let $t' = \chi^A \in \mathcal{T}_{QR}$. Since $\{c_{l-m}, \ldots, c_i\}$ contains at least two crossovers, $c_i \geq c_{\ell-m} + n_1 u + m_1 a + u - a$. Therefore, $c_i - c_{\ell-m} \geq \max\{k, n_1 u + m_1 a + u - a\} = c(t')$. Let $y = y_R - \{p_R\} + \{t'\} + y_{end}$. Tiling y covers all channels, and has cost

$$c(y) = c(y_R - \{p_R\}) + c(t') + c(y_{end}) \leq c_f - c_0.$$

If $p \in \mathcal{P}_Q$, let $p' = p + \chi^{\{v\}}$ where v is any node in $A \cap R$. Then $c(p') = c(p) + u$. Also, let $t' = \chi^{A-\{v\}}$. If $m_1 > 1$, then $c(t') = \max\{k, n_1 u + (m_1 - 1)a + u - a\}$. If $m_1 = 1$, then $c(t') = \max\{k, n_1 u\} \leq \max\{k, n_1 u + (m_1 - 1)a + u - a\}$. Since $p \in \mathcal{P}_Q$ only if w is in Q, c_i was chosen so that $c_i \geq c_{\ell-m} + k + u$. Furthermore, the channels $\{c_{\ell-m}, \ldots, c_i\}$ are covered by $n_1 + 1$ nodes in Q, m_1 nodes in R, and contain at least three crossovers. Hence, $c_i \geq c_{\ell-m} + (n_1 - 1)u + (m_1 - 2)a + 3u$. Hence, $c_i - c_{\ell-m} - u \geq \max\{k, n_1 u + m_1 a + u - 2a\} \geq c(t')$. Let $y = y_R - \{p_R\} + \{p'\} + y_{end} - \{p\} + \{t'\}$. The tiling y covers all channels, has a patch of the right type, and has cost

$$\begin{aligned}
c(y) &= c(y_R - \{p_R\}) + c(t') + c(y_{end}) + (c(p') - c(p)) \\
&\leq (c_{\ell-m} - c_0) + (c_i - c_{\ell-m} - u) + (c_f - c_i) + u \\
&= c_f - c_0.
\end{aligned}$$

If $p \in \mathcal{P}_{QR}^{big}$, let $V_Q = V(p) \cap Q$, $V_R = V(p) \cap R$, $n^p = |V_Q|$, and $m^p = |V_R|$. Let B_Q and B_R be the subsets of Q are R, respectively, that contain those nodes of weight two in p. Let $n_2^p = |B_Q|$ and $m_2^p = |B_R|$. Then $c(p) = (n^p + n_2^p)u + (m_2^p - 1)a + \max\{k, m^p a\}$.

We form two new tiles, $t' = \chi^{V_Q} + \chi^{V_R} \in \mathcal{T}_{QR}$ and $t'' = \chi^A \in \mathcal{T}_{QR}$, as well as a patch $p' = \chi^{B_Q} + \chi^{B_R} \in \mathcal{P}_{QR}$. Then $c(t') = \max\{k, n^p u + m^p a + u - a\}$, $c(t'') = \max\{k, n_1 u + m_1 a + u - a\}$ and $c(p') = n_2^p u + (m_2^p - 1)a$. We now verify that $c(t') + c(p') - c(p) \leq u - a$.

If $k \geq n^p u + m^p a + u - a$, then $k \geq m^p a$ and $c(t') + c(p') - c(p) = k + (n_2^p u + (m_2^p - 1)a) - ((n^p + n_2^p)u + (m_2^p - 1)a + k) = -n_2^p u$. Otherwise, $c(t') + c(p') - c(p) \leq (n^p u + m^p a + u - a) + (n_2^p u + (m_2^p - 1)a) - ((n^p + n_2^p)u + (m_2^p - 1)a + m^p a) = u - a$. Hence, $c(t') + c(p') - c(p) \leq u - a$.

Note that $p \in \mathcal{P}_{QR}^{big}$ only if $w \in Q$. Using the same argument that was used for the case $p \in \mathcal{P}_Q$, it can be shown that $c_i - c_{\ell-m} - u \geq \max\{k, n_1 u + m_1 a + u - 2a\}$. Therefore, $c_i - c_{\ell-m} - u + a \geq \max\{k, n_1 u + m_1 a + u - a\} = c(t'')$.

Let $y = y_R - \{p_R\} + y_{end} - \{p\} + \{t'\} + \{t''\} + \{p'\}$. The tiling y covers all channels, has a patch of the right type, and has cost

$$\begin{aligned} c(y) &= c(y_R - \{p_R\}) + c(y_{end}) + (c(t') + c(p') - c(p)) + c(t'') \\ &\leq (c_{\ell-m} - c_0) + (c_f - c_i) + (u - a) + (c_i - c_{\ell-m} - u + a) \\ &= c_f - c_0. \end{aligned}$$

Case 2B. y_Q contains tiles other than p_Q.

By Lemma 5.2, patch p_Q covers channels $\{c_{j-n}, \ldots, c_{j-1}\}$, and these channels are all assigned to nodes in Q, so $j - n \geq \ell$. By induction there exists a tiling y_{end} that covers channels $\{c_{j-n}, \ldots, c_f\}$.

Let p be the patch of y_{end}. As previously, $V_Q = V(p) \cap Q$, $V_R = V(p) \cap R$, $n^p = |V_Q|$ and $m^p = |V_R|$. If $p \in \mathcal{P}_{QR}^{big}$, then we also define B_Q, B_R, n_2^p and m_2^p as in Case 2A. Let t be a tile from $y_Q - \{p_Q\}$, let $V_t = V(t)$ and $n^t = |V_t|$. Then $c(t) = \max\{k, n^t u\}$.

In the following table, we show how we will combine p_R, p and t into a new tile t' and a new patch p'.

Case	Condition	Tile t'	Patch p'
(1)	$p \in \mathcal{P}_{QR}^{big}$		
(1.1)	(1) and $V_t \cap V_Q = \emptyset$	$p + t$	p_R
(1.2)	(1) and $V_t \cap V_Q \neq \emptyset$	$p + \chi^{V_t - V_Q}$	$\chi^{V_t \cap V_Q} + p_R$
(2)	$p \in \mathcal{P}_{QR}$	$t + \chi^{V_R}$	$\chi^{V_Q} + p_R$
(3)	$p \in \mathcal{P}_Q$	t	$p + p_R$

Case	Cost $c(t')$	$t' \in$	Cost $c(p')$	$p' \in$
(1.1)	$\max\{k, (n^p + n^t)u\} + \max\{k, m^p a\}$ $+n_2^p u + m_2^p a + u - a$	$\mathcal{T}_{QR}^{big}$	$c(p_R)$	$\mathcal{P}_R$
(1.2)	$\max\{k, \|V_t \cup V_Q\|u\} + \max\{k, m^p a\}$ $+n_2^p u + m_2^p a + u - a$	$\mathcal{T}_{QR}^{big}$	$\|V_t \cap V_Q\|u + c(p_R)$	$\mathcal{P}_{QR}$
(2)	$\max\{k, n^t u + m^p a + u - a\}$	$\mathcal{T}_{QR}$	$n^p u + c(p_R)$	$\mathcal{P}_{QR}$
(3)	$c(t)$	$\mathcal{T}_Q$	$n^p u + c(p_R)$	$\mathcal{P}_{QR}$

Table 5: Combining patches

In all cases, we form the new tiling

$$y = y_R - \{p_R\} + y_Q - \{p_Q\} - \{t\} + y_{end} - \{p\} + \{t'\} + \{p'\}.$$

It can be verified, using the table, that in all cases, $c(p') + c(t') - c(t) - c(p_R) - c(p) \le u$.

Consider Case (1.1). Since $p \in \mathcal{P}_{QR}$, $c(p) = (n^p + n_2^p)u + (m_2^p - 1)a + \max\{k, m^p a\}$. Hence, $c(t') - c(p) = \max\{k, (n^p + n^t)u\} - n^p u + u \le c(t) + u$. Since $c(p') = c(p_R)$, we have $c(p') + c(t') - c(t) - c(p_R) - c(p) \le u$.

In Case (1.2), we again have $c(p) = (n^p + n_2^p)u + (m_2^p - 1)a + \max\{k, m^p a\}$. Hence, $c(t') - c(p) + c(p') - c(p_R) = \max\{k, |V_t \cup V_Q|u\} - n^p u + u + |V_t \cap V_Q|u \le c(t) + u$. Hence, $c(p') + c(t') - c(t) - c(p_R) - c(p) \le u$.

In Case (2), $c(p) = n^p u + (m^p - 1)a$. Therefore, $c(p') - c(p) - c(p_R) = -(m^p - 1)a$. Furthermore, $c(t') - c(t) \le u + (m^p - 1)a$. Hence, $c(p') + c(t') - c(t) - c(p_R) - c(p) \le u$.

In Case (3), $c(p) = (n^p - 1)u$. Therefore, $c(p') - c(p) - c(p_R) = u$. Since $c(t') = c(t)$, we have $c(p') + c(t') - c(t) - c(p_R) - c(p) \le u$.

Therefore, in each case, y covers all channels, the patch of y is of the right type, and

$$\begin{aligned}
c(y) &= c(y_R) + c(y_Q - \{p_Q\}) + c(y_{end}) + (c(p') + c(t') - c(t) - c(p) - c(p_R)) \\
&\le (c_{\ell-1} - c_0) + (c_{j-n} - c_\ell) + (c_f - c_0) + u \\
&= c_f - c_0 - (c_\ell - c_{\ell-1} - u) \\
&\le c_f - c_0
\end{aligned}$$

Example 5.8 *We will find a tiling of G in Example 5.1, restricted to the channels 12 to 79. The tiling y_R will cover channels in $\{12, 13\}$, while y_Q covers channels $\{15, 17, 22\}$. Then $y_R = \chi^{\{r_1,r_3\}}$ and $y_Q = \chi^{\{q_1,q_2\}} + \chi^{\{q_1\}}$. Since 12 is assigned to a node in R and y_Q contains a tile other than p_Q, this examples falls under Case 2B.*

We need a tiling that covers channels 22 through 79. We can used tiling y_4 from Example 5.5. Note that $c(y_4) = 50$. The tiling y_4 has patch $p = \chi^{\{q_1,q_5,r_1,r_4\}} \in P_{QR}$. We, therefore, need to consider Case (2).

By definition, $V_Q = \{q_1, q_5\}$ and $V_R = \{r_1, r_4\}$. The tiling y_Q contains only one tile besides the patch, so $t = \chi^{\{q_1,q_2\}}$. Therefore, $t' = \chi^{\{q_1,q_2\}} + \chi^{\{r_1,r_4\}} = \chi^{\{q_1,q_2,r_1,r_4\}}$ and $p' = \chi^{\{q_1,q_5\}} + \chi^{\{r_1,r_3\}} = \chi^{\{q_1,q_3,r_1,r_3\}}$.

Therefore, the final tiling is $y_5 = y_4 - \{p\} + \{t'\} + \{p'\} = y_4 - \chi^{\{q_1,q_5,r_1,r_4\}} + \chi^{\{q_1,q_2,r_1,r_4\}} + \chi^{\{q_1,q_5,r_1,r_3\}}$. Hence, $c(y_5) = c(y_4) - (2 \cdot 2 + (2-1) \cdot 1) + \max\{5, 2 \cdot 2 + 2 \cdot 1 + 2 - 1\} + (2 \cdot 2 + (2-1) \cdot 1) = 51 - 5 + 7 + 5 = 58$

Example 5.9 *In Example 5.7, it was shown that $y_6 = \chi^{\{q_1,q_4\}} + y_5 - \{p\} + p'$, where p is the patch of y_5 and p' is the patch of y_6, is a tiling that covers the graph given in Example 5.1. We now give the tiling explicitly.*

¿From Example 5.8, we obtain $y_6 = \chi^{\{q_1,q_4\}} + y_4 - \{p\} + \chi^{\{q_1,q_2,r_1,r_4\}} + p'$ *where* p *is the patch of* y_4*.*

¿From Example 5.5, we have $y_6 = \chi^{\{q_1,q_4\}} + \chi^{\{q_1,r_1,q_3\}} + y_3 - \{p\} + \chi^{\{q_1,q_2,r_1,r_4\}} + p'$ *where* p *is the patch of* y_3*.*

¿From Example 5.4 we have $y_6 = \chi^{\{q_1,q_4\}} + \chi^{\{q_1,r_1,q_3\}} + \chi^{\{q_3,q_4\}} + \chi^{\{q_2,r_1,q_5\}} + y_2 - \{p\} + \chi^{\{q_1,q_2,r_1,r_4\}} + p'$ *where* p *is the patch of* y_2*.*

¿From Example 5.3 we have $y_6 = \chi^{\{q_1,q_4\}} + \chi^{\{q_1,r_1,q_3\}} + \chi^{\{q_3,q_4\}} + \chi^{\{q_2,r_1,q_5\}} + \chi^{\{q_4\}} + \chi^{\{q_4,q_1\}} + \chi^{\{q_3,q_1,r_2,r_3,r_4,r_1\}} + y_1 - \{p\} + \chi^{\{q_1,q_2,r_1,r_4\}} + p'$ *where* p *is the patch of* y_1*.*

Finally, from Example 5.2, $y_6 = \chi^{\{q_1,q_4\}} + \chi^{\{q_1,r_1,q_3\}} + \chi^{\{q_3,q_4\}} + \chi^{\{q_2,r_1,q_5\}} + \chi^{\{q_4\}} + \chi^{\{q_4,q_1\}} + \chi^{\{q_3,q_1,r_2,r_3,r_4,r_1\}} + \chi^{\{r_2,r_3,r_4\}} + \chi^{\{r_1,r_2,r_3\}} + \chi^{\{q_1,q_2,r_1,r_4\}} + p'$*, where* $p' = \chi^{\{r_3,r_4\}} + \chi^{\{q_1,q_5,r_1,r_3\}} + \chi^{\{q_5\}} \in \mathcal{P}_{QR}^{big}$*. This tiling does indeed cover all channels, has a patch of the required type, and has a cost of* $c(y_6) = 69 \leq 79$*, as required.*

This completes the proof. □

6 Conclusions

We have described how a new general method of obtaining lower bounds for the channel assignment problems, when applied to the specific example of nested cliques, leads to a complete family of lower bounds which include almost all known clique-bounds, are easy to compute, and give improved results when applied to an example from the literature.

Further work should address the computational issues related to lower bounds. A computational study comparing the performance of tile cover bounds to the TSP bound and the network flow bound on a number of realistic CAP instances would be a valuable addition to this theoretical analysis.

It is also an interesting question whether the tile cover approach can be used to obtain good channel assignments. Knowledge about which lower bound is most restrictive for any particular instance could be used to determine which tiles were most suited to build the best assignment.

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Appendix A

In this Appendix we give the proof of Lemma 5.2. The lemma as stated in this paper differs slightly from the version presented in [10], since a slightly extended version was needed for its use in the proof of Lemma 5.3. Also, the proof of this lemma gives the flavour of proof of Lemma 5.3, and may therefore by useful to the reader.

Lemma 5.2[10]
Let G be a clique with co-site constraint k and edge constraint u. Let Q be the node set of G, and let the tile set $\mathcal{T}_Q$ and patch set $\mathcal{P}_Q$ be as defined above. Then for any channel assignment of (G, w) of span s there exists a tile cover $y \in \mathbb{Z}^{\mathcal{T}_Q \cup \mathcal{P}_Q}$, which contains exactly one patch, p, of (G, w) with cost at most s. Moreover, the support of p consists of the nodes that receive the last $|V(p)|$ channels of the assignment.

Proof. Let f be a channel assignment of (G, w) of span s, using channels $\{c_0, c_1, \ldots, c_f\}$, where $c_0 < c_1 < \ldots < c_f$. Let $\mu = \lfloor \frac{k}{u} \rfloor$. We will construct the required tile cover tile by tile.

For all j, $0 \leq j \leq f$, let the partial tile cover y_j denote a collection of tiles (no patches) such that $c(y_j) \leq c_j - c_0$ and y_j covers channels $\{c_0, \ldots, c_{j-1}\}$. We start the construction of the tile cover with the empty tile collection $y_0 = 0$, so $c(y_0) = 0 = c_0 - c_0$. Next, supposing that we already have a partial tile cover y_j, we proceed to construct a new family $y_{j'}$ for some higher value $j' > j$.

(i) If any node of G receives a channel $c_{j'}$ such that $c_j + k \leq c_{j'} < c_j + (\mu + 1)u$, then since G is a clique and because channels on neighbouring nodes have to differ by at least u, no node in G has a channel in the interval $[c_j + \mu u, c_j + k)$. Let A be the set that contains all nodes of G with a channel in the range $[c_j, c_j + \mu u)$, and let the tile $t = \chi^A$; t covers all channels between c_j and $c_{j'-1}$. Since A can contain at most μ nodes, t has cost k. Let $y_{j'} = y_j + \{t\}$, then $c(y_{j'}) = c(y_j) + c(t) \leq c_j - c_0 + k \leq c_{j'} - c_0$.

(ii) If (i) fails and $c_f \geq c_j + (\mu + 1)u$, then no node can have *two* channels from the range $[c_j, \ldots, c_j + (\mu + 1)u)$, because of the requirement that channels on the same node have to differ by at least k. Let $c_{j'}$ be the least channel such that $c_{j'} \geq c_j + (\mu + 1)u$. Let A be the set that contains all nodes of G with a channel in the range $[c_j, c_j + (\mu + 1)u)$, and let tile $t = \chi^A$; t covers all channels between c_j and $c_{j'-1}$. Since A can contain at most $\mu + 1$ nodes, t has cost at most $(\mu + 1)u$. Let $y_{j'} = y_j + \{t\}$, then $c(y_{j'}) = c(y_j) + c(t) \leq c_j - c_0 + (\mu + 1)u \leq c_{j'} - c_0$.

(iii) If (i) fails and $c_f < c_j + (\mu + 1)u$, then we conclude the construction by adding one patch, $p = \chi^A$, where A is the set which consists of the nodes of G with a channel in the range $[c_j, c_f]$. Since $c_f < c_j + (\mu + 1)u$, $|A| \leq \mu + 1$, and because

(i) fails, no node can have two channels from $\{c_j, \ldots, c_f\}$. Take $y = y_j + \{p\}$. Now $c_f \geq c_j + (|A| - 1)u$, and thus $c(y) = c(y_j) + c(p) = c_j - c_0 + (|A| - 1)u \leq c_f - c_0 = s$. Therefore, y is the desired tile cover. $\square$

In fact, it is possible to strengthen the lemma and show that for a clique with uniform co-site constraint k and edge constraint u, for any tile cover of cost s, there exists a channel assignment that covers the same weights of span at most s. By using tile covers of this kind, bounds can be obtained from a polyhedral analysis in the same way as was done in the previous sections. For a complete discussion of this case, see [8], and for a synopsis see [10].