



# Infinite limits of copying models of the web graph

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Technical Report CS-2003-04

April, 2003

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# INFINITE LIMITS OF COPYING MODELS OF THE WEB GRAPH

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ABSTRACT. Several stochastic models were proposed recently to model the dynamic evolution of the web graph. We study the infinite limits of the stochastic processes proposed to model the web graph when time goes to infinity. We prove that deterministic variations of the so-called copying model can lead to several non-isomorphic limits. Some models converge to the infinite random graph  $R$ , while the convergence of other models is sensitive to initial conditions or minor changes in the rules of the model. We explain how limits of the copying model of the web graph share several properties with  $R$  that seem to reflect known properties of the web graph.

## 1. INTRODUCTION

The web may be viewed as a directed graph with nodes the static HTML web pages, and directed edges representing the links between web pages. This graph is commonly referred to as *the web graph*; it is an example of a massive network, with several billion nodes. Several interesting properties were observed in the web graph: in particular, the in- and out-degrees seem to satisfy a power law degree distribution, the web graph is *small world*, which means that it has high clustering and low diameter, and it is locally dense while globally sparse. (See [20] for a survey of properties of the web graph.) Another interesting property of the web graph is that it is *evolutionary*: nodes appear and disappear with time. Throughout this paper, we will consider the simple, undirected version of the web graph. (The reason for this is that the structural results we present are best described in graphs where edges have no orientation; we consider the directed case as the next step in our study.)

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1991 *Mathematics Subject Classification*. 05C80, 68R10, 94C15.

*Key words and phrases*. web graph, evolving copying model, preferential attachment model, the infinite random graph, adjacency property, inexhaustible graph, bipartite cores, Hamilton paths.

The authors gratefully acknowledges support from NSERC research grants, and from a MITACS grant.

Owing to its massive and dynamic nature, several authors have suggested statistical models which capture certain properties of the web graph. These models are loosely based on classical random graphs, first introduced by Erdős and Rényi. If  $n$  is a positive integer, and  $0 < p < 1$  is a fixed real number, then a random graph  $G(n, p)$  has  $n$  nodes, and there is an edge between two nodes with probability  $p$ . The graphs  $G(n, p)$  have several drawbacks as models of the web graph. For example, the degree distribution of random graphs is binomial, rather than satisfying a power law; further, the number of nodes is static. These drawbacks maybe overcome by making the model dynamic, and by assigning different probabilities to various nodes. Two models that take these approaches are the *preferential attachment model* of Barabási and Albert [3] and the *evolving copying model* of Kumar et al [19]. In the preferential attachment model, we start with a small base graph. At each time step, we create a new node, say  $u$ , and draw its edges according to a predetermined distribution. In particular, node  $u$  is joined to an existing node  $v$  with probability proportional to  $\deg(v)$ . In the evolving copying model, we start with a small base graph. At each time step, we create a new node,  $u$ . Choose an existing node  $v$  uniformly at random (u.a.r.). For each edge  $vw$ , with probability  $1 - p$ , add the edge  $uw$ . Hence, the neighbourhood of the new node  $u$  will be a subset of the neighbourhood of the existing node  $v$ . A slight variation is the *evolving copying model with error*, where with probability  $p$ , an edge is added between  $u$  and an existing node chosen u.a.r.

The first analysis of the long-term behaviour of these models has been made, for example, by Aiello, Chung, Lu [1, 2], Cooper, Frieze [10], and by Kumar *et al.* [19]. Power law degree distributions were proven to exist in both the preferential attachment [3] and evolving copying models [19]. Many bipartite cliques were shown to exist in evolving copying models [19], mirroring the abundance of so-called “cyber-communities” measured by bipartite cliques in the web graph (as reported in [18]).

Our motivating question is: *what are the resulting graphs like if we allow these stochastic processes to continue indefinitely?* We attempt to answer this question in the case of the copying model of the web graph. The resulting graphs are infinite, and are *limits* (that is, unions of chains) of finite web graph models. On the surface, the study of infinite graphs may appear to have no connection with the study of a finite experimental graph such as the web graph. However, limits of web graph models have certain fractal and other properties which correlate with known data on the web graph obtained by various web crawls. (See Theorems 7 and 9.)

If we consider limits of the  $G(n, p)$  graphs, then the resulting graph will almost surely be isomorphic to the *infinite random graph*, written  $R$ . The graph  $R$  is the unique (up to isomorphism) countable graph satisfying the following *existentially closed* or *e.c.* adjacency property.

**e.c. property:** A graph  $G$  is e.c. if for each pair of finite disjoint subsets  $X$  and  $Y$  of nodes of  $G$ , there exists a node  $z_{X,Y} \in V(G) \setminus (X \cup Y)$  that is joined to each node of  $X$  and to no node of  $Y$ .

For more on  $R$ , the reader is directed to the excellent survey [9]. The graph  $R$  may be viewed as the limit of an evolutionary process. For this, let  $R_0$  be a single node; assume that  $R_n$  is defined and contains  $R_0$ . Enumerate all of the finite subsets of nodes of  $R_n$ , and extend each of these subsets, in all possible ways, by new nodes not in  $R_n$ . The resulting graph we call  $R_{n+1}$ , and the union of the chain  $(R_n : n \in \omega)$  is an e.c. graph that is isomorphic to  $R$ . The preceding construction of  $R$  serves as a template for what follows, where we will consider infinite graphs grown by certain evolutionary processes. Our results show that graphs grown in this way have many properties in common with  $R$ , although they are usually not isomorphic to  $R$ . See Sections 3 and 4.

## 2. ADJACENCY PROPERTIES AND LIMITS

All the graphs we consider are undirected, simple, and have a countable number of nodes. We use the notation  $\omega$  for the set of natural numbers considered as an ordinal, and  $\aleph_0$  is the cardinality of  $\omega$ . The cardinality of the real numbers is written  $2^{\aleph_0}$ . If  $S \subseteq V(G)$ , then  $G \upharpoonright S$  is the subgraph induced by  $S$ . If  $G$  is an induced subgraph of  $H$ , then we write  $G \leq H$ . The graph  $G \uplus H$  is the disjoint union of  $G$  and  $H$ . If  $y$  is a node of  $G$ , then  $N(y) = \{z : yz \in E(G)\}$  is the *neighbour set* of  $y$  in  $G$ . The *closed neighbour set* of  $y$ , written  $N[y]$ , is the set  $N(y) \cup \{y\}$ . If  $x$  is a node of  $G$ , then the graph  $G - x$  is the graph  $G \upharpoonright (V(G) \setminus \{x\})$ . If  $S \subseteq V(G)$ , then  $G - S$  is defined similarly. A node is *isolated* if it has no neighbours, and it is *universal* if it is joined to all nodes except itself.

To study the limits of web graph models, we consider graphs satisfying various deterministic adjacency properties that are more general than the e.c. property described in the Introduction. We note that determinism has been used by other authors in the study of the web graph; for instance, see [4] for a study of deterministic scale-free networks. Let  $X$  and  $Y$  be disjoint finite sets of nodes in a graph  $G$ . We say that the node  $z_{X,Y} \in V(G) \setminus (X \cup Y)$  is *correctly joined to  $X$*  and  $Y$ , if  $z_{X,Y}$  is joined to each node of  $X$  and no node of  $Y$ .

**Property (A):** A graph  $G$  has property (A) if for each node  $y$  of  $G$ , for each finite  $X \subseteq N[y]$ , and each finite  $Y \subseteq V(G) \setminus X$ , there exists a node  $z_{X,Y} \neq y$  which is correctly joined to  $X$  and  $Y$ .

**Property (B):** Property (B) is defined similarly to Property (A), except that  $N[y]$  is replaced by  $N(y)$ .

For a fixed  $n \in \omega$ , properties  $(A,n)$  and  $(B,n)$  are defined analogously to (A) and (B) respectively, but the node  $z$  may be joined to at most  $n$  other nodes. More precisely:

**Property (A, $n$ ):** A graph  $G$  has property  $(A,n)$  for some  $n \in \omega$  if for each node  $y$  of  $G$ , for each finite  $X \subseteq N[y]$ , for each finite  $Y \subseteq V(G) \setminus X$ , and for each set  $U \subseteq V(G) \setminus (N[y] \cup Y)$  with cardinality at most  $n$ , there is a node  $z_{X,Y,U} \neq y$  correctly joined to  $X \cup U$  and  $Y$ .

**Property (B, $n$ ):**

Again, Property  $(B,n)$  is defined similarly to Property  $(A,n)$ , except that  $N[y]$  is replaced by  $N(y)$ .

Note that property (A) is just  $(A,0)$ , and (B) is just  $(B,0)$ . We sometimes say that a graph with property  $(\mathcal{P})$ , where  $\mathcal{P}$  is one of A or B, is a  $(\mathcal{P})$  graph.

The adjacency properties (A) and (B) are inspired by the evolving copying model of the web graph, while properties  $(A,n)$  and  $(B,n)$  are inspired by the evolving copying model with error. The idea (that will be made precise in the sequel) is that as time goes to infinity, any extension that is made with positive probability is almost surely true in the limit.

Is there anything that can be said about the structure of graphs with these adjacency properties? How do these graphs compare and contrast with  $R$ , and with the actual web graph? In this section and the next, we will attempt to answer these questions. A first observation is that we have the following chain of logical implications (for all integers  $n \geq 1$ ),

$$\text{e.c.} \Rightarrow (A,n) \Rightarrow (B,n) \Rightarrow (A) \Rightarrow (B).$$

Our first theorem gives insight into the structure of graphs with (A). A graph is  $\aleph_0$ -universal if it embeds all countable graphs as induced subgraphs. For example, it well-known that  $R$  is  $\aleph_0$ -universal; see [9], for example.

**Theorem 1.** *Let  $G$  satisfy (A). Then for all  $y \in V(G)$ ,  $G \upharpoonright N(y) \cong R$ . In particular,  $G$  is  $\aleph_0$ -universal.*

*Proof.* Fix  $y \in V(G)$ . By remarks in the Introduction, it is enough to show that  $N = G \upharpoonright N(y)$  is e.c. For this, fix  $X$  and  $Y$ , disjoint subsets of  $V(N)$ . By (A) there is a node  $z_{X,Y}$  of  $G$  that is joined to each node of  $\{y\} \cup X$ , but is not joined nor equal to any of the nodes in  $Y$ . Then  $z_{X,Y}$  is a node of  $N$ , and therefore,  $N$  satisfies the e.c. property.  $\square$

In the next theorem, we see that if  $n \geq 1$ , then the adjacency properties (A, $n$ ) and (B, $n$ ) are in fact *equivalent* to the e.c. property. We find this surprising, since then adding a single extra “random link” gives a deterministic conclusion. As we will see in Theorem 5, however, there are uncountably many non-isomorphic countable graphs with (A) or (B). Thus, Theorems 2 and 5 seem to contrast the evolving copying and evolving copying with error models.

**Theorem 2.** *Fix  $n > 0$  an integer. If  $G$  has (B, $n$ ), then  $G$  is isomorphic to  $R$ .*

*Proof.* To show that  $G$  is isomorphic to  $R$ , we need only show that  $G$  is e.c.: for all finite disjoint subsets  $C$  and  $D$  of nodes of  $G$ , there is a node  $z$  of  $V(G) \setminus (C \cup D)$  that is joined to each node of  $C$  and to no node of  $D$ .

*Case 1.*  $|C| \leq n$ .

Choose any node  $y \notin C \cup D$ . Let  $X = C \cap N(y)$ ,  $Y = D$ , and  $U = C \setminus N(y)$ . By property (B, $n$ ), there exists a node  $z_{X,U,Y}$  correctly joined to  $X \cup U = C$  and  $Y = D$ .

*Case 2.*  $|C| > n$ .

We can then write  $C$  as  $W_1 \cup \dots \cup W_r$ , where the  $W_i$  are pairwise disjoint and have cardinality at most  $n$ . Now, as in Case 1, choose a node  $y$  not in  $C \cup D$ . Let  $X = N(y) \cap W_1$ ,  $Y = D$ , and let  $U = W_1 \setminus N(y)$ . By the (B, $n$ ), there is a node  $x_1 \notin X \cup Y$  such that  $W_1 \subseteq N(x_1)$ . By the (B, $n$ ) property with  $X = W_1$ ,  $Y$  empty, and  $U = W_2$ , there is a node  $x_2$  not in  $X \cup Y$  that is joined to all of  $W_1 \cup W_2$ . Proceeding inductively, we can find a node  $x_r$  that is joined to all of the nodes in  $C \cup D$ . As then  $C \cup D \subseteq N(x_r)$ , by a final application of (B, $n$ ), there is a node  $z$  correctly joined to  $C$  and  $D$ .  $\square$

Because of Theorem 2, we will restrict our attention to graphs satisfying properties (A)=(A,0) and (B)=(B,0) for the rest of the paper. If  $(G_n : n \in \omega)$  is a sequence of graphs with  $G_n \leq G_{n+1}$ , then define

$$\lim_{n \rightarrow \infty} G_n = \bigcup_{n \in \omega} G_n;$$

we call  $\lim_{n \rightarrow \infty} G_n$  the *limit* of the sequence  $(G_n : n \in \omega)$ . We say that  $(G_n : n \in \omega)$  is a *chain* of graphs.

A graph  $G$  has an (A)-*constructing sequence* if there is a chain of graphs  $(G_n : n \in \omega)$  such that  $\lim_{n \rightarrow \infty} G_n = G$ , which has the following properties. (A (B)-*constructing sequence* is defined analogously, using  $N(y)$  rather than  $N[y]$ .)

- (1)  $G_0$  is a finite graph.
- (2) For each integer  $n > 1$ ,  $G_{n+1}$  is obtained from  $G_n$  by one application of process (P1) followed by a finite number (possibly zero) applications of process (P2), where (P1) and (P2) are defined as follows:
  - (P1) For each node  $y$  of  $G_n$ , and for each finite  $X \subseteq N[y]$ , a new node  $z_X \notin V(G_n)$  is added whose neighbours in  $V(G_n)$  are exactly the nodes of  $X$ . We say that  $z_X$  *extends*  $X$ . We say that *all subsets of closed neighbour sets in  $V(G_n)$  are extended in all ways*.
  - (P2) For a finite fixed  $X \subseteq V(G_n)$ , a new node  $z_X \notin V(G_n)$  is added whose neighbours in  $V(G_n)$  are exactly the nodes of  $X$ .

The graph  $G$  is then  $\lim_{n \rightarrow \infty} G_n$ . We refer to the graphs  $G_n$  as *time-steps* in the evolution of  $G$ . If (P2) is never used at any time-step, then we say that the corresponding construction sequence is *pure*; otherwise, the constructing sequence is *mixed*. A graph  $G$  is *pure* if it has a pure constructing sequence. Otherwise, we say that  $G$  is *mixed*. In a mixed constructing sequence, the nodes added in (P2) are called *extra* nodes. We note that  $R$  has both an (A)- and (B)-constructing sequence where (P2) is used at each time-step. (We leave the details as an exercise.)

We note that a graph  $G$  formed by an (A)-constructing sequence has property (A). The converse also holds.

**Lemma 1.** *Let  $G$  be a graph with  $V(G) = \omega$ , and fix a finite induced subgraph  $H$ . If  $\mathcal{P}$  is A or B, and if  $G$  has property ( $\mathcal{P}$ ), then  $G$  has an ( $\mathcal{P}$ )-constructing sequence  $(G_n : n \in \omega)$  with  $G_0 = H$ .*

*Proof.* Assume that  $\mathcal{P} = A$ . By relabelling nodes if necessary, we may assume that  $0 \in V(G_0)$ . We will show how to construct a chain of graphs  $(G_n : n \in \omega)$  with the following property: for each  $n \in \omega$ ,  $n \in V(G_n)$  and  $G_n$  is a finite induced subgraph of  $G$ . Note that from this property, it follows that  $\lim_{n \rightarrow \infty} G_n = G$ .

Let  $G_0 = H$ . Inductively, assume that  $G_n$  is defined and finite. For each node  $y$  of  $G_n$ , and each subset  $X \subseteq N[y]$ , let  $Y = V(G_n) \setminus X$ . Since  $G$  has property (A), there exists a node  $z_X \in V(G) \setminus V(G_n)$  that is joined to all nodes in  $X$ , and none in  $Y$ . Let  $V'$  be the set of such nodes  $z_X$ , exactly one for each subset  $X \subseteq N[y]$  in  $G_n$ . Define  $G_{n+1}$  to be the finite subgraph of  $G$  induced by  $V(G_n) \cup V' \cup \{n+1\}$ . It is clear

that, for all  $n \in \omega$ ,  $G_n \leq G_{n+1}$ , so  $(G_n : n \in \omega)$  is a chain. Adding each node  $z_X$  in  $V'$  to  $G_n$  corresponds to an application of process (P1). If the node  $n + 1$  is not in  $V'$ , then adding  $n + 1$  to  $G_n$  corresponds to one application of process (P2).

The proof for property (B) is similar, and so is omitted.  $\square$

**Corollary 1.** *If  $\mathcal{P}$  is either A or B, then the following are equivalent.*

- (1) *The graph  $G$  has property ( $\mathcal{P}$ ).*
- (2) *The graph  $G$  has an ( $\mathcal{P}$ )-constructing sequence.*

### 3. MANY MODELS

Our main results concern graphs with properties (A) and (B). We first show that properties (A) and (B) are not  $\aleph_0$ -categorical; that is, there are many non-isomorphic graphs that satisfy these properties.

A *homomorphism* from  $G$  to  $H$  is a mapping  $f : V(G) \rightarrow V(H)$  that *preserves edges*; more precisely,  $xy \in E(G)$  implies that  $f(x)f(y) \in E(H)$ . We usually write  $f : G \rightarrow H$  or just  $G \rightarrow H$ . If  $G \rightarrow H$  and  $H \rightarrow G$ , then we say that  $G$  and  $H$  are *homomorphically equivalent*, and write  $G \leftrightarrow H$ . See [17] for more on graph homomorphisms.

**Theorem 3.** *Let  $H$  be a finite graph. Let  $G$  be an infinite pure (B) graph with a pure (B)-constructing sequence  $(G_n : n \in \omega)$ , where  $G_0 = H$ . Then  $H \leftrightarrow G$ .*

*Proof.* As  $H \leq G$ , we have that  $H \rightarrow G$ . To show that  $G \rightarrow H$ , we construct a homomorphism by induction on  $n \in \omega$ . Let  $f_0$  be the identity map on  $G_0 = H$ . Suppose that  $f_n : G_n \rightarrow H$  is a homomorphism extending  $f_0$ . Let  $z \in V(G_{n+1}) \setminus V(G_n)$ . Then by the definition of (P1) there exist a node  $y \in V(G_n)$  and a subset  $X$  of  $N(y)$ , so that  $z$  is only joined to nodes of  $X$ . So in  $G_{n+1}$ ,  $N(z) \subseteq N(y)$ . We label this node  $y$  as  $y_z$ . Since  $f$  is a homomorphism,  $f_n(y_z) \neq f_n(x)$  for all  $x \in N(y_z)$ . Hence, we may map  $z$  to  $f_n(y_z)$  and preserve edges. Therefore, the map  $f_{n+1} : G_{n+1} \rightarrow H$  defined by

$$f_{n+1}(z) = \begin{cases} f_n(z) & \text{if } z \in V(G_n); \\ f_n(y_z) & \text{else,} \end{cases}$$

is a homomorphism. The map  $F : G \rightarrow H$  defined by  $\bigcup_{n \in \omega} f_n$  is a homomorphism.  $\square$

The following Corollary is immediate from Theorem 3.

**Corollary 2.** *For a fixed finite graph  $H$ , let  $G(H)$  be an infinite pure (B) graph with a pure (B)-constructing sequence  $(G_n : n \in \omega)$  such that*



$G_0 = H$ . Then the following hold:

- (1)  $\chi(G(H)) = \chi(H)$  and  $\omega(G(H)) = \omega(H)$ .
- (2) If  $H$  and  $H'$  are not homomorphically equivalent, then  $G(H) \not\cong G(H')$ .

We note that there are infinite families of non-homomorphically equivalent finite graphs; see [5]. Hence, there are at least  $\aleph_0$  many non-isomorphic pure (B) graphs; see Theorem 6. This contrasts with the situation for pure (A) graphs.

**Theorem 4.** *There is a unique pure (A) graph, up to isomorphism.*

We will defer the proof of Theorem 4 to Section 4, since our proof will make heavy use of the inexhaustibility property which is discussed there. If  $n$  is a positive integer, then a graph  $G$  is *n-existentially closed* or *n-e.c.* if each  $n$ -subset  $S$  of  $V(G)$  can be extended in all ways. Hence, a graph  $G$  is e.c. if it is  $n$ -e.c. for all positive integers  $n$ . It is well-known that for every constant  $p \in (0, 1)$ , and fixed  $n$  a positive integer, almost all finite random graphs with edges chosen independently with probability  $p$  are  $n$ -e.c. We use this property to give the maximum possible cardinality of non-isomorphic mixed graphs with property (A).

**Theorem 5.** *There are  $2^{\aleph_0}$  many non-isomorphic infinite mixed graphs with property (A).*

*Proof.* Fix  $n \geq 5$ . Let  $G_0 = C_{n+1}$ , the chordless cycle on  $n + 1$  nodes. Assume that  $G_i$  is defined and finite. To form  $G_{i+1}$ , first apply process (P1) and extend all subsets of closed neighbour sets of  $G_i$  to form  $G'_{i+1}$ . Then apply process (P2) a finite number of times by extending all  $n$ -subsets of nodes of  $G'_{i+1}$  to form  $G_{i+1}$ . Define  $G(n) = \lim_{n \rightarrow \infty} G_i$ . The graph  $G(n)$  has Property (A) with  $(G_i : i \in \omega)$  an (A)-constructing sequence, and is clearly  $n$ -e.c. Note that  $G(n)$  is mixed, since (P2) is used in the constructing sequence. However, there is no node in  $G(n)$  that is joined to *each node of*  $G_0$ , so  $G(n)$  is not  $(n+1)$ -e.c. To see this, we proceed by induction on  $i$ . Assume that there is no node in  $G_i$  joined to all of  $G_0$ . In  $G_{i+1}$ , the nodes that are added to  $G_i$  are of two types: 1) extending subsets of closed-neighbour sets in  $G_i$ , or 2) extending arbitrary  $n$ -subsets in  $G_i$ . The nodes of type 2 can never be joined to all the  $n + 1$  nodes of  $G_0$ . Now consider nodes of type 1. Assume, to obtain a contradiction, that  $V(G_0) \subseteq N[y]$  for some  $y$  in  $G_i$ . Then  $y$  cannot equal an element of  $G_0$ , as  $G_0$  contains no universal nodes. Hence,  $V(G_0) \subseteq N(y)$  which contradicts our induction hypothesis. Therefore, there is no type 1 node in  $V(G_{i+1}) \setminus V(G_i)$  joined to each node of  $G_0$ .

Since any  $n$ -e.c. graph, where  $n \geq 2$ , is connected, it follows that each graph  $G(n)$  is connected. Now, let  $X$  be an infinite subset of  $\omega$ , listed as  $X = \{n_i : i \in \omega\}$ . Define

$$G(X) = \bigsqcup_{i \in \omega} G(n_i).$$

Hence, the connected components of  $G(X)$  are the  $G(n_i)$ . Then  $G(X)$  satisfies (A), since property (A) is preserved by taking disjoint unions, as is readily verified. Let  $Y$  be an infinite subset of  $\omega$  with  $X \neq Y$ . Let  $n \in X \setminus Y$ . Then  $G(X)$  contains a connected component that is  $n$ -e.c. but not  $(n+1)$ -e.c. However, there is no such connected component in  $G(Y)$ ; thus,  $G(X) \not\cong G(Y)$ . As there are  $2^{\aleph_0}$  many infinite subsets of  $\omega$ , there are  $2^{\aleph_0}$  many non-isomorphic (A) graphs. As there is a unique isomorphism type of pure (A) graph by Theorem 4, there are  $2^{\aleph_0}$  many non-isomorphic mixed (A) graphs.  $\square$

**Theorem 6.** *There are  $2^{\aleph_0}$  many non-isomorphic infinite mixed graphs with property (B) but not (A). There are exactly  $\aleph_0$  many non-isomorphic infinite pure graphs with property (B) but not (A).*

*Proof.* Let  $G$  be a pure (B) graph, with a (B)-constructing sequence  $(G_i : i \in \omega)$  so that  $G_0 = K_n$ , for a fixed  $n \geq 4$ . Since  $G$  is pure,  $G_{i+1}$  is obtained from  $G_i$  only by process (P1), for each positive integer  $i$ . By Corollary 2,  $\chi(G) = \chi(G(K_n)) = n$ .

**Claim:**  $G = G' \uplus \overline{K_{\aleph_0}}$ , where  $G'$  is a connected graph with  $\chi(G') = n$ .

To see this, note that  $G_{i+1}$  was constructed by adding nodes joined to some or no nodes of  $G_i$ . Suppose that  $G_i = G_1(i) \uplus G_2(i)$ , where  $(G_1(i))$  is connected and  $G_2(i)$  is independent. Let  $G_1(i+1)$  be those nodes in  $G_{i+1}$  that are joined to *some* node of  $G_i$ , and let  $G_2(i+1)$  be the nodes in  $G_{i+1}$  that are not joined to *any* node of  $G_i$ . By definition of (P1), the new nodes of  $G_{i+1}$  are either joined to the neighbourhood of a node in  $G_i$ , and thus must be connected to  $G_1(i)$ , or they are independent, and hence they are part of  $G_2(i+1)$ . Therefore,  $(G_1(i+1))$  is connected and contains  $G_1(i)$ , and  $G_2(i+1)$  forms an independent set in  $G_{i+1}$  and contains  $G_2(i)$ . Moreover,  $G_2(i+1)$  properly contains  $G_2(i)$ , because according to process (P1), for every node  $z$  of  $G_i$ , for the choice  $X = \emptyset$ , a new independent node  $z_X$  is added. Note that  $G_{i+1} = G_1(i+1) \uplus G_2(i+1)$ . Note also that  $G_0 = K_n \uplus H_0$ , where  $H_0$  is the empty graph. Let  $\lim_{i \rightarrow \infty} (G_1(i)) = G'$  and let  $\lim_{i \rightarrow \infty} (G_2(i)) = H$ . The graph  $G'$  is connected, as each graph  $G_1(i)$  is connected, and  $H$  is independent, since each  $G_2(i)$  is independent. Since the cardinality of

$G_2(i)$  is strictly increasing,  $H \cong \overline{K_{\aleph_0}}$ . Also, since  $G(i) = G_1(i) \uplus G_2(i)$  for each  $i$ , and  $\lim_{i \rightarrow \infty} G_i = G$ ,  $G = G' \uplus H$ . As  $G_0 \leq G' \leq G$ , it is immediate that  $G'$  has chromatic number  $n$ .

Now let  $\Omega = \{K_n : n \geq 4\}$ . For a fixed infinite  $X \subseteq \Omega$ , define a graph  $G(X)$  as follows. Let  $X = \{K_{n_i} : i \in \omega\}$ . As in Corollary 2, let  $G(K_{n_i})$  be the graph obtained from  $G_0 = K_{n_i}$  by a pure (B)-constructing sequence. By setting  $G(K_{n_i})$  to be the graph  $G$  defined earlier in the proof, we obtain that  $G(K_{n_i}) = G'(n_i) \uplus \overline{K_{\aleph_0}}$ , where  $G'(n_i)$  is a connected graph with  $\chi(G'(i)) = n_i$ . Define  $G(X)$  as  $\biguplus_{i \in \omega} G(K_{n_i})$ . Then  $G(X)$  has property (B), but note that  $G(X)$  cannot have (A) by Theorem 1, since the chromatic number of each connected component is finite. Now if  $X, Y$  are infinite subsets of  $\Omega$  with  $X \neq Y$ , then suppose that  $K_n \in X \setminus Y$ . By the Claim, there is no component in  $G(Y)$  with chromatic number  $n$ , so  $G(X) \not\cong G(Y)$ . Hence, there are  $2^{\aleph_0}$  many non-isomorphic (B) graphs.

Let  $G$  be a pure (B) graph with pure (B)-constructing sequence  $(G_i : i \in \omega)$ . It is not hard to see that  $G$  is determined up to isomorphism by the finite graph  $G_0$ . As there are only  $\aleph_0$  many non-isomorphic choices for  $G_0$ , there are at most  $\aleph_0$  non-isomorphic pure (B) graphs. By Corollary 2 there are at least  $\aleph_0$  non-isomorphic pure (B) graphs, at least one with chromatic number  $n$ , for each  $n \in \omega$ . Hence, there are exactly  $\aleph_0$  many non-isomorphic pure (B) graphs, and therefore, by the last sentence in the previous paragraph, there are  $2^{\aleph_0}$  many non-isomorphic mixed (B) graphs.  $\square$

We note that the infinite random graph  $R$  has property (A) (and therefore (B)), but has neither a pure (A)- nor (B)-constructing sequence. To see this in the case of property (A), we note that any pure (A) graph is disconnected. Let  $G$  be a pure (A) graph with pure (A)-constructing sequence  $(G_i : i \in \omega)$ . At each time step  $G_n$ , where  $n > 1$ , at least two isolated nodes are introduced. For a fixed  $n > 1$ , call two such nodes in  $G_n$   $u$  and  $v$ . An inductive argument shows that in the following time-steps  $G_r$ , with  $r > n$ ,  $u$  and  $v$  remain in different components of  $G_r$ . Hence, there are at least two connected components in  $G$ . However, since  $R$  is e.c., it is connected of diameter 2. Therefore,  $R$  cannot have a pure (A)-constructing sequence. We proved in Corollary 2 that any pure (B) graph has finite chromatic number, and so  $R$  cannot have a pure (B)-constructing sequence.

#### 4. FRACTAL AND OTHER PROPERTIES

A graph  $G$  is *inexhaustible* if for all  $x \in V(G)$ , we have that  $G - x \cong G$ . The graph  $R$  is inexhaustible, as are the infinite complete and

null graphs. For more on inexhaustible graphs, the reader is directed to [6] and [15]. We prove that the same property holds for graphs satisfying properties (A) or (B). Inexhaustibility is an example of a *fractal property* of graphs: an inexhaustible graph is resilient under node deletion, a property observed in the actual web graph in [11]. For more on fractal properties of graphs, the reader is directed to [7].

**Theorem 7.** *If  $G$  is a fixed graph with property (B), then  $G$  is inexhaustible.*

*Proof.* Let  $(G_n : n \in \omega)$  be a (B)-constructing sequence for  $G$ . If  $n$  is a positive integer, then a set of nodes in  $G_n$  is called *n-special* if it includes nodes of  $V(G_n) \setminus V(G_{n-1})$ . We introduce the following notation for subsets of  $V(G)$ . Let  $S_1$  be the set of nodes added to  $G_0$  in time step 1 by extending sets of nodes of  $G_0$ . In general, in  $G_r$ , let  $S_r$  be the set of nodes extending  $G_{r-1}$ . Let  $S_{1,1} = S_1$ . In  $S_2$ , there are nodes  $S_{2,1} \subseteq S_2$  extending  $G_0$  as  $S_1$  does, and nodes  $S_{2,2} \subseteq S_2$  extending 1-special sets of nodes. Note that  $S_{2,1} \cup S_{2,2} = S_2$  and  $S_{2,1} \cap S_{2,2} = \emptyset$ . In general, in  $S_r$  we define a finite sequence  $(S_{r,i} : 1 \leq i \leq r)$  of sets of nodes partitioning  $S_r$ , with each  $S_{r,i}$  consisting of the nodes that extend the  $(i-1)$ -special sets of  $G_{r-1}$ . In particular,  $S_{r,r}$  is the *only* set extending  $(r-1)$ -special sets of nodes. If  $1 \leq i \leq r-1$ , then the nodes in the set  $S_{r,i}$  extend the same subsets that  $S_{r-1,i}$  does.

Let  $G_{\infty,0} = G_0$ . To define  $G_{\infty,1}$ , we form disjoint sets of nodes  $(S_{\infty,1,i} : i \in \omega)$ , each disjoint from  $V(G_{\infty,0})$  and of the same cardinality as  $S_{1,1}$ , and let

$$V(G_{\infty,1}) = V(G_{\infty,0}) \cup \bigcup_{i \in \omega} S_{\infty,1,i}.$$

Now let each  $S_{\infty,1,i}$  extend  $G_0$  as  $S_{1,1}$  does. More precisely, the subgraph of  $G_{\infty,1}$  induced by  $V(G_0)$  and  $S_{\infty,1,i}$  is isomorphic to  $G_1$ . Moreover,  $\bigcup_{i \in \omega} S_{\infty,1,i}$  is an independent set in  $G_{\infty,1}$ . It follows from the definition that  $G_{\infty,1}$  contains infinitely many subgraphs isomorphic to  $G_1$ .

Assume that  $G_{\infty,r}$  is defined, countable, and contains infinitely many subgraphs isomorphic to  $G_r$ . To define  $G_{\infty,r+1}$ , form disjoint sets of nodes  $(S_{\infty,r,i} : i \geq r)$ , each disjoint from  $V(G_{\infty,0})$  and of the same cardinality as  $S_{r,r}$ . Let

$$V(G_{\infty,r}) = V(G_{\infty,r-1}) \cup \bigcup_{i \geq r} S_{\infty,r,i}.$$

Let each of the  $S_{\infty,r,i}$  extend one of the subgraphs of  $G_{\infty,r}$  isomorphic to  $G_r$  as  $S_{r,r}$  does, and let  $\bigcup_{i \geq r} S_{\infty,r,i}$  form an independent set in  $G_{\infty,r+1}$ .

Clearly,  $G_{\infty,r+1}$  contains infinitely many subgraphs isomorphic to  $G_r$ . See Figure 1.

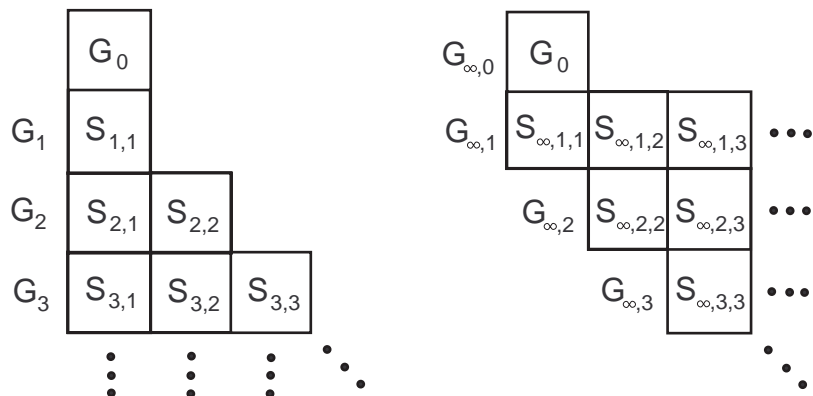


FIGURE 1. The graphs  $G$  and  $G_\infty$ .

**Claim 1:**  $G \cong G_\infty$ .

We define a mapping  $f : G \rightarrow G_\infty$  as follows. The map  $f$  sends  $G_0$  to  $G_{\infty,0}$  via the identity map. For each  $i, j \in \omega \setminus \{0\}$ , map  $S_{i,j}$  isomorphically onto  $S_{\infty,j,i}$ . (Hence, “columns” in  $G$  are mapped to “rows” in  $G_\infty$ ; see Figure 1.) It is straightforward to see that  $f$  is an isomorphism. Claim 1 follows.

Now, we prove that  $G_\infty$  is inexhaustible. Fix  $x \in V(G_\infty)$ .

*Case 1.*  $x \notin V(G_{\infty,0})$ .

Let  $m$  be the largest non-negative integer so that  $G_{\infty,m}$  does not contain  $x$ . Let  $f_m : G_{\infty,m} \rightarrow G_{\infty,m}$  be the identity mapping.

**Claim 2:** For all  $r > m$ ,  $G_{\infty,r} \cong G_{\infty,r} - x$  via an isomorphism  $f_r$  extending  $f_{r-1}$ .

We proceed by induction on  $r$ . We use the *back-and-forth method* to define  $f_r$ , which is a two player game of perfect information played in countably many steps on two graphs  $X_0, X_1$ . The players are named the *duplicator* and the *spoiler*. (The names come from the facts that the duplicator is trying to show the graphs are alike, while the spoiler is trying to show they are different.) A move consists of a choice of a node from either graph, and the spoiler makes the first move. The players take turns choosing nodes from the  $V(X_i)$ , so that if one player chooses a node from  $V(X_i)$ , the other must choose a node of  $V(X_{i+1})$  (the indices are mod 2). The game begins in our case with a fixed isomorphism  $f$  between two induced subgraphs  $Y_0$  and  $Y_1$  of  $X_0$  and  $X_1$ , respectively. Players cannot choose previously chosen nodes, or

nodes in a  $Y_i$ . After  $n$  rounds, this gives rise to a list of nodes  $Y_0 \cup \{a_i : 1 \leq i \leq n\}$  from  $X_0$  and  $Y_1 \cup \{b_i : 1 \leq i \leq n\}$  from  $X_1$ . The duplicator wins if for every  $n \geq 1$ , the subgraph induced by  $Y_0 \cup \{a_i : 1 \leq i \leq n\}$  is isomorphic to the subgraph induced by  $Y_1 \cup \{b_i : 1 \leq i \leq n\}$  via an isomorphism extending  $f$  mapping  $a_i$  to  $b_i$ , for every  $1 \leq i \leq n$ . Otherwise, the spoiler wins. From this it follows that the duplicator has a winning strategy if and only if  $X_0$  and  $X_1$  are isomorphic via an isomorphism extending  $f$ . See [8] for more on the back-and-forth method.

We let

$$X_0 = G_{\infty,r}, X_1 = G_{\infty,r} - x, f = f_{r-1}, Y_0 = G_{\infty,r-1}, Y_1 = G_{\infty,r-1} - x.$$

Going forward, suppose that the spoiler chooses  $y$  in  $G_{\infty,r}$ , where  $y$  is not in  $G_{\infty,r-1}$ . We will assume that  $y$  is a node added at time-step  $G_r$  by process (P1) (the argument for process (P2) is similar and so is omitted). Then  $y$  extends some set  $A$  in  $N(z)$ , for some  $A$  and  $z$  in  $G_{\infty,r-1}$ . Let  $A' = f_{r-1}(A)$ , and  $z' = f_{r-1}(z)$  in  $G_{\infty,r-1} - x$ . Hence, there is a  $y'$  in  $G_{\infty,r}$  extending  $A'$  as  $y$  extends  $A$ . If  $y' \neq x$ , then the duplicator chooses this node. If  $y' = x$ , then the duplicator may choose any of the infinitely many nodes in  $G_{\infty,r} - x$  that also extends  $A'$  as  $y'$  does. Going back is similar. Since any two nodes in  $V(G_{\infty,r}) \setminus V(G_{\infty,r-1})$  are non-joined, Claim 2 follows.

The map  $\bigcup_{r \in \omega} f_r : G_{\infty} \rightarrow G_{\infty} - x$  is an isomorphism.

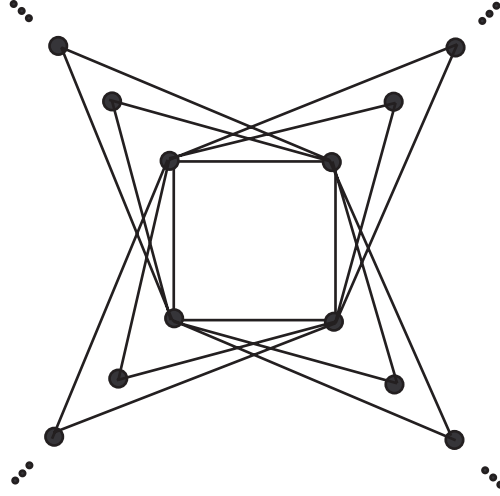
*Case 2.*  $x \in V(G_{\infty,0})$ .

Given the finite graph  $G_{\infty,0}$ , define the infinite graph  $G'_{\infty,0}$  as follows. For each node  $y$  of  $G_{\infty,0}$ , add infinitely many new pairwise non-joined nodes  $y_i$  with the property that  $y_i$  has the same neighbours in  $G_{\infty,0}$  as  $y$  does. (We think of the  $y_i$  as nodes extending  $N(y)$ . Hence,  $G'_{\infty,0} \leq G_{\infty,1}$  in  $G_{\infty}$ .) It is straightforward to see that  $G'_{\infty,0}$  is inexhaustible. (See Figure 2 for a depiction of the graph  $C'_4$ .)

Define a new graph  $G'_{\infty}$  that is constructed as  $G_{\infty}$  was, but beginning with  $G'_{\infty,0}$ , rather than  $G_{\infty,0}$ .

**Claim 3:**  $G'_{\infty} \cong G_{\infty}$ .

To prove Claim 3, we first prove that  $G'_{\infty,1}$  and  $G_{\infty,1}$  are isomorphic by extending the identity mapping  $g_0$  between  $G_{\infty,0} \leq G'_{\infty,1}$  and  $G_{\infty,0} \leq G_{\infty,1}$ . Suppose that the spoiler chooses a node  $y'$  in  $V(G'_{\infty,1}) \setminus V(G_{\infty,0})$ . Consider the case when  $y'$  was added by process (P1). (The argument for the case when  $y'$  is added by process (P2) is similar, and so is omitted.) The duplicator can respond with a node  $y \in V(G_{\infty,1})$  joined to  $A \subseteq V(G_{\infty,0})$ . Going back is similar. Since no two nodes in  $V(G_{\infty,1}) \setminus V(G_{\infty,0})$  or in  $V(G'_{\infty,1}) \setminus V(G_{\infty,0})$  are joined, as

FIGURE 2. The graph  $C'_4$ .

we noted in Case 1, the duplicator can win. Using similar arguments, we obtain, for each  $r \in \omega$ , isomorphisms  $g_r : G'_{\infty,r} \rightarrow G_{\infty,r}$  so that if  $r \geq 1$ , then  $g_r \upharpoonright G'_{\infty,r-1} = g_{r-1}$ . The map

$$\bigcup_{r \in \omega} g_r : G'_{\infty} \rightarrow G_{\infty}$$

is an isomorphism.

To finish Case 2, it is therefore sufficient to show that  $G'_{\infty}$  is inexhaustible. Choose  $y \in V(G'_{\infty})$ . If  $y \in V(G'_{\infty,0})$ , since  $G'_{\infty,0}$  is inexhaustible, there is an isomorphism  $g_0 : G'_{\infty,0} \rightarrow G'_{\infty,0} - y$ . As in Case 1, extend  $g_0$  to isomorphisms  $g_r : G'_{\infty,r} \rightarrow G'_{\infty,r} - y$ , for all  $r \geq 0$ , by back-and-forth. The map

$$\bigcup_{r \in \omega} g_r : G'_{\infty} \rightarrow G'_{\infty} - x$$

is an isomorphism. If  $y \notin V(G'_{\infty,0})$ , then proceed as in Case 1.  $\square$

Recall that a graph which has an (A)-constructing sequence is *pure* if at each time step, process (P1) is used; it is *mixed* otherwise. One may ask whether there are many non-isomorphic pure (A) graphs. Using Theorem 7, we prove that the answer is negative, and thereby prove Theorem 4 of Section 3 above. This is in stark contrast to the situation for mixed (A) graphs, as proved in Theorem 5. If  $G_0$  is a finite graph, we use the notation  $\uparrow G_0$  for the unique (up to isomorphism) graph that results by applying the (P1) process for property (A) recursively

to  $G_0$ . It follows that every pure (A) graph is of the form  $\uparrow G_0$  for some finite graph  $G_0$ . It is not hard to see that  $\uparrow (G \uplus H) \cong \uparrow G \uplus \uparrow H$ .

*Proof of Theorem 4:* It is enough to show that if  $G_0$  and  $H_0$  are finite graphs, then  $\uparrow G_0 \cong \uparrow H_0$ . For this let  $(G_i : i \in \omega)$  and  $(H_i : i \in \omega)$  be (A)-constructing sequences for  $G = \uparrow G_0$  and  $H = \uparrow H_0$ , respectively. As  $H$  is  $\aleph_0$ -universal by Theorem 1,  $G \leq H$ . In particular, there is some  $n \in \omega$  so that  $G_0 \leq H_n$ . Delete from  $H$  all the finitely many nodes in  $S = V(H_n) \setminus V(G_0)$ . At time-step  $n + 1$ , we are left with a copy of  $G_1$  extending  $G_0$  (as it is extended in  $G$ ), and finitely many isolated nodes, say  $m$  of them (that were either joined to no node of  $H_n$  in  $H$ , or were joined only to nodes that we deleted). Hence,  $H_{n+1} - S \cong G_1 \uplus \overline{K_m}$ . Since  $G_1$  extends  $G_0$ , we have that  $H - S \cong \uparrow (G_0 \uplus \overline{K_m}) \cong \uparrow G_0 \uplus \uparrow \overline{K_m}$ .

At each time-step  $r$  in the construction of  $G$  ( $r > 0$ ), nodes with no edges to  $G_{r-1}$  are added to  $G_r$ . These nodes give rise to connected components of  $G$  of the form  $\uparrow K_1$ . Hence,  $G$  contains infinitely many connected components of the form  $\uparrow K_1$ . It follows that  $G$  is isomorphic to the graph  $J$ , which consists of the disjoint union of  $\uparrow G_0$  and infinitely many disjoint copies of  $\uparrow K_1$ .

As  $H$  is inexhaustible by Theorem 7, it follows that  $H - S \cong H$ . But then

$$H \cong H - S \cong \uparrow (G_0 \uplus \overline{K_m}) \cong \uparrow G_0 \uplus \uparrow \overline{K_m} \cong J \cong G,$$

since  $\uparrow \overline{K_{\aleph_0}} \uplus \uparrow \overline{K_m} \cong \uparrow \overline{K_{\aleph_0}}$ . □

The unique isomorphism type of Theorem 4 we name  $R_N$ , since it is locally isomorphic to  $R$ . We do not know much about  $R_N$ . The infinite random graph  $R$  is *indivisible*: whenever the nodes of  $R$  are coloured red or blue, there is a monochromatic induced subgraph isomorphic to  $R$ . For more on indivisible graphs, see [12, 15]. A graph without this property is *divisible*. A graph  $G$  so that  $R \leq G$  is necessarily indivisible, since  $R$  is itself indivisible. Therefore, by Theorem 1, a graph with property (A), such as  $R_N$ , is indivisible. It is not hard to see that a graph with at least one edge and with finite chromatic number is divisible, so by the proof of Theorem 5 for (B), there are examples of graphs with (B) that are divisible.

A *ray* is an infinite path that extends indefinitely in one direction; a *double ray* is an infinite path that extends indefinitely in two directions. A *one-way Hamilton path* is a spanning subgraph that is a ray, while a *two-way Hamilton path* is a spanning subgraph that is a double ray. The graph  $R$  contains one- and two-way Hamilton paths.



**Theorem 8.** *If  $G$  has property (B), then the connected components of  $G$  have one and two-way Hamilton paths. In particular,  $G$  has a 1-factor.*

*Proof.* Let  $G'$  be a fixed connected component of  $G$ . We prove that  $G'$  has a one-way Hamilton path; the existence of a two-way Hamilton path is similar. Without loss of generality, let  $V(G') = \omega$ .

Define  $P_0$  be the subgraph induced by  $\{0\}$ . Assume that there is a path  $P(n)$  in  $G'$  containing the nodes  $\{0, \dots, n\}$ , and that the nodes of  $P(n)$  are  $x_1, \dots, x_s$ . If the node  $n+1$  equals some  $x_i$ , then let  $P(n+1) = P(n)$ . Otherwise, assume that  $n+1$  is not a node in  $P(n)$ . As  $G'$  is connected, the node  $n+1$  is connected by a path  $Q$  to some node  $x_i$  of  $P(n)$ . Let  $Q$  be the path  $y_1 y_2 \dots y_{t-1} y_t$ , where  $y_1 = n+1$  and  $y_t = x_i$ . As  $x_{i+1}$  and  $y_{t-1}$  are in  $N(x_i)$ , by (B) there is a node  $z_1$  in  $G$ , and hence, in  $G'$ , that is joined to both  $x_{i+1}$  and  $y_{t-1}$ , and is not a node of  $P(n)$  nor  $Q$ .

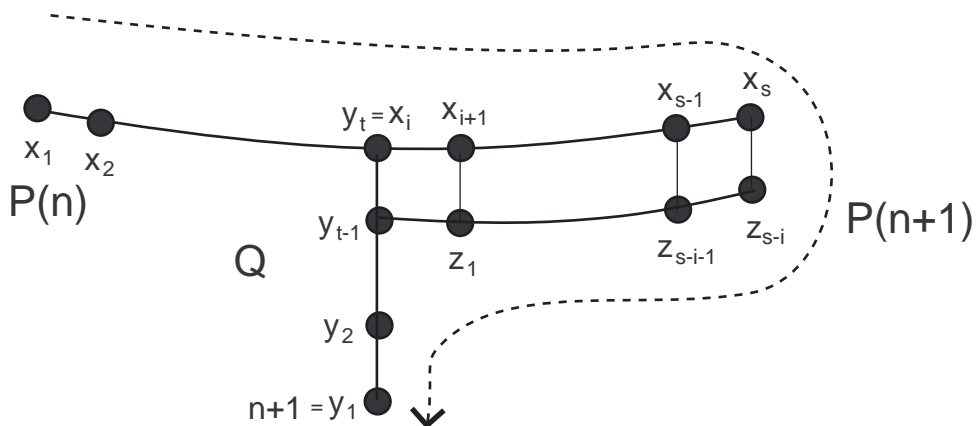


FIGURE 3. A Hamilton path containing  $\{0, \dots, n+1\}$ .

Proceeding inductively, we obtain a path of nodes  $z_1 z_2 \dots z_{s-i}$  in  $G'$  so that  $z_{j+1}$  is joined to both  $z_j$  and  $x_{i+j+1}$ . Then the path  $P(n)$  followed by the path  $z_{s-i} \dots z_2 z_1$  is a path  $P(n+1)$  containing  $\{0, \dots, n+1\}$ . The desired Hamilton path of  $G'$  is  $\lim_{n \rightarrow \infty} P(n)$ . See Figure 3.  $\square$

A *bipartite core* is a graph on  $i+j$  nodes that contains at least one bipartite clique  $K_{i,j}$  as a subgraph. Bipartite cores arise in the web graph as representing communities of users who have pages on some common topic; see [19, 20]. The findings of [18] reveal that the web graph contains at least several hundred thousand such cores. Many bipartite cliques almost surely arise in the evolving copying model of

[19], so we should expect that graphs satisfying (B) would contain many bipartite cliques. The infinite random graph  $R$  contains every countable bipartite graph as an induced subgraph, so there are many such bipartite cores in graphs with the e.c. property, or with (A) by Theorem 1. The following theorem proves a similar result for graphs with (B). The infinite bipartite graph with each vertex class infinite is written  $K_{\aleph_0, \aleph_0}$ .

**Theorem 9.** *If  $G$  is a graph with (B) so that  $G_0$  has at least one edge, then  $K_{\aleph_0, \aleph_0} \leq G$ . In particular, there are infinitely many node-disjoint bipartite cores in a graph satisfying (B).*

*Proof.* Let  $S_1$  be some edge of  $G$  (by the fact that  $G_0$  has edges, we know that  $G$  also has edges). Assume that  $S_n \leq G$  is isomorphic to  $K_{n,n}$  with  $S_1 \leq S_n$ . Suppose that  $S_n$  has nodes  $A \cup B$ , with  $A = \{x_i : 1 \leq i \leq n\}$ ,  $B = \{y_i : 1 \leq i \leq n\}$  consisting of independent sets. By property (B), there is a node  $x_{n+1}$  not in  $A \cup B$  that is joined to each node of  $B$ . Since  $A \cup \{x_{n+1}\} \subseteq N(y_1)$ , by (B), there is a node  $y_{n+1}$  not in  $A \cup B \cup \{x_{n+1}\}$  that is joined to each node of  $A \cup \{x_{n+1}\}$ . Then the graph  $S_{n+1}$  consisting of  $S_n$  along with  $x_{n+1}$  and  $y_{n+1}$  is a bipartite clique strictly containing  $S_n$ . The graph  $H = \bigcup_{n \geq 1} S_n \leq G$  is isomorphic to  $K_{\aleph_0, \aleph_0}$ .  $\square$

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